Symmetric generating functions, permutation statistics and basic hypergeometric series

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November 11, 2022

Outline:

1. Permutation statistics (Mahonian, Eulerian, Stirling)
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1. Permutation statistics

Permutation statistics

A permutation statistic is a function from the set of permutations to the set of positive integers. There are three classical kinds of permutation statistics: Mahonian, Eulerian and Stirling statistics.

Let S_n be the set of permutations of $[n]$. For a given permutation $\pi = \pi_1\pi_2\ldots\pi_n \in S_n$, define

$$des(\pi) = |\{i \in [n-1] : \pi_i > \pi_{i+1}\}|,$$

be the number of descents of π. Define

$$maj(\pi) = \sum_i i \cdot \chi(\pi_i > \pi_{i+1})$$

be the major index of π. Here $\chi(A) = 1$ if A is true; otherwise $\chi(A) = 0$. Furthermore, define

$$lmax(\pi) = |\{i \in [n] : \pi_i > \pi_j \text{ for all } j < i\}|$$

be the number of left-to-right maxima.

Permutation statistics

Any statistic whose distribution over S_n equals the one of des/maj/lmax is called an Eulerian/Mahonian/Stirling statistic.

The well-known Eulerian polynomial is

$$A_n(t) = \sum_{\pi \in S_n} t^{des(\pi)+1}, \text{ which satisfies}$$

$$\sum_{m \geq 0} m^n t^m = \frac{A_n(t)}{(1-t)^{n+1}}.$$

For the Mahonian statistic maj, we have

$$\sum_{\pi \in S_n} t^{maj(\pi)} = \prod_{i=1}^{n} \frac{1-t^i}{1-t}.$$

For the Stirling statistic lmax,

$$\sum_{\pi \in S_n} t^{lmax(\pi)} = t(t+1)\cdots(t+n-1).$$

Permutation statistics

Examples of Mahonian/Eulerian/Stirling statistics:

Mahonian: maj, imaj, inv.

$$\textit{imaj}(\pi) = \mathrm{maj}(\pi^{-1}),$$
$$\textit{inv}(\pi) = |\{(i,j) : \pi_i > \pi_j \text{ and } i < j\}|.$$

Eulerian: des, ides, exc.

$$\textit{ides}(\pi) = \mathrm{des}(\pi^{-1}),$$
$$\textit{exc}(\pi) = |\{i \in [n] : \pi_i > i\}|.$$

Stirling: lmax, lmin, rmax.

$$\textit{lmax}(\pi) = |\{i \in [n] : \pi_i > \pi_j \text{ for all } j < i\}|,$$

and lmin (left-to-right minima), rmax (right-to-left maxima) are defined similarly.

2. Symmetric refined generating functions

Symmetric refined generating functions

An elegant symmetric generating function for the joint distribution of (des,ides) was found by Carlitz, Roselle and Scoville (JCTA, 1966):

$$\sum_{n=0}^{\infty}\sum_{\pi\in S_n}\frac{u^{des(\pi)}x^{ides(\pi)}t^n}{(1-u)^{n+1}(1-x)^{n+1}}=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{x^{n-1}u^{k-1}}{(1-t)^{kn}}.$$

It is clear that (des, ides, maj, imaj)π =(ides, des, imaj, maj)π^{-1}, so the generating function for the joint distribution of (des, ides, maj, imaj) must be bi-symmetric.

The corresponding generating function formula was subsequently discovered by Garsia and Gessel (Adv. Math., 1979), employing the theory of P-partitions and its nice amenability to record permutation statistics.

Symmetric refined generating functions

$$\frac{1}{(1-u_1)(1-u_2)} \sum_{n=0}^{\infty} \sum_{\pi \in S_n} u_1^{des(\pi)} u_2^{ides(\pi)} q_1^{maj(\pi)} q_2^{imaj(\pi)} t^n$$

$$\times \frac{1}{(1-u_1 q_1)\cdots(1-u_1 q_1^n)(1-u_2 q_2)\cdots(1-u_2 q_2^n)}$$

$$= \sum_{k_1 \geq 0} \sum_{k_2 \geq 0} u_1^{k_1} u_2^{k_2} \prod_{i \leq k_1} \prod_{i \leq k_2} \frac{1}{1 - t q_1^i q_2^j}.$$

Proof ingredient: (P, ω)-partition. Given a poset P, define a labeling of P to be a bijection $\omega : P \to [n]$, then a map $\sigma \to N$ is called a (P, ω)-partition if (1) σ is order-preserving, i.e., when $s \leq t$ in P, $\sigma(s) \leq \sigma(t)$; and (2) if $s \leq t$ in P and $\omega(s) > \omega(t)$, then $\sigma(s) < \sigma(t)$.

This notion generalizes compositions (when P is trivial) and partitions (when P is a vertical line).

Symmetric refined generating functions

If we add Stirling statistics such as lmax, lmin and rmax, what would the generating functions look like and whether the symmetric distribution is preserved? The purpose of our paper (arXiv: 2210.08789) is to answer these questions.

$$\mathcal{G}(t;x,u,v,q,z) := \sum_{n=1}^{\infty} t^n \sum_{\pi\in S_n} \frac{u^{des(\pi)} x^{ides(\pi)} q^{lmin(\pi)} v^{rmax(\pi)} z^{lmax(\pi)}}{(1-u)^n(1-x)^n}.$$

Through a bijection from permutations to inversion sequences by Baril and Vajnovszki (2017), we have an equivalent form:

$$\mathcal{G}(t;x,u,v,q,z) := \sum_{n=1}^{\infty} t^n \sum_{s\in I_n} \frac{u^{asc(s)} x^{dist(s)} q^{max(s)} v^{rmin(s)} z^{zero(s)}}{(1-u)^n(1-x)^n}.$$

An inversion sequence $s = (s_1, s_2, \ldots, s_n)$ is a sequence of non-negative integers satisfying $0 \le s_i < i$ for all i.

Symmetric refined generating functions

Theorem 1 (J. 2022)

The generating function for the joint distribution of a quadruple (*des*, *ides*, *rmax*, *lmin*) of Euler-Stirling statistics on permutations equals (with $r = 1 - t$)

$$\mathcal{G}(t; x, u, v, q, 1) = \frac{vt}{1-x} \sum_{n=1}^{\infty} \frac{qx - 1 + (1-q)r^{n-1}}{x - r^n} \prod_{i=1}^{n-1} \frac{u(x - r^i - xvt)((1-qt)r^{i-1} - 1)}{(r^i - 1)(x - r^i)}$$

$$\times \left(1 - \frac{ut(q-1)}{x-1} \sum_{n=1}^{\infty} \frac{(x - 1 - xv)r^{n-1} + xv}{r^n - 1} \right.$$

$$\left. \times \prod_{i=1}^{n-1} \frac{u(x - r^i - xvt)((1-qt)r^{i-1} - 1)}{(r^i - 1)(x - r^i)} \right)^{-1},$$

and $\mathcal{G}(t; x, u, v, q, 1) = \mathcal{G}(t; x, u, q, v, 1) = \mathcal{G}(t; u, x, q, v, 1)$, namely, (*des*, *ides*, *rmax*, *lmin*), (*des*, *ides*, *lmin*, *rmax*), (*ides*, *des*, *rmax*, *lmin*) are all equidistributed over the set S_n.

Symmetric refined generating functions

Theorem 2 (J. 2022)

The generating function for the joint distribution of a triple (des,ides,rmax), or (des,ides,lmin), (ides,des,lmin) or (ides,des,rmax) of Euler-Stirling statistics on permutations equals

$$\mathcal{G}(t;x,u,v,1,1) = \sum_{n=1}^{\infty} \frac{vtu^{n-1}}{(1-t)^n - x} \prod_{i=1}^{n-1} \frac{x-(1-t)^i - xvt}{x-(1-t)^i}$$

$$= \sum_{n=1}^{\infty} \frac{vt(1-t)^{n-1}}{u^n(x(1-t)^{n-1}-1)} \prod_{i=1}^{n} \frac{1-x(1-t)^{i-1}}{x(1-t)^{i-1}(vt-1)+1}$$

and $\mathcal{G}(t;x,u,v,1,1) = \mathcal{G}(t;u,x,1,v,1) = \mathcal{G}(t;x,u,1,v,1) = \mathcal{G}(t;u,x,v,1,1)$.

An analytic proof of the fact $\mathcal{G}(t;x,u,v,1,1) = \mathcal{G}(t;u,x,v,1,1)$ follows from a transformation formula of non-terminating basic hypergeometric ${}_4\phi_3$ series by J. and Schlosser 2020.

Symmetric refined generating functions

Corollary 1 (J. 2022)

The generating function for the joint distribution of (des,ides) on permutations equals

$$\sum_{n=0}^{\infty}\sum_{\pi\in S_n}\frac{u^{des(s)}x^{ides(\pi)}t^n}{(1-u)^{n+1}(1-x)^{n+1}}$$

$$=\sum_{n=1}^{\infty}\frac{t(1-t)^{n-1}(u-1)^{-1}}{u^n(1-x(1-t)^{n-1})(1-x(1-t)^n)}+\frac{1}{(u-1)(x-1)}$$

$$=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{x^{n-1}u^{k-1}}{(1-t)^{kn}}.$$

This recovers the classical result by Carlitz, Roselle and Scoville (1966). The second equation can also be proved by manipulations on formal power series.

Symmetric refined generating functions

Theorem 3 (J. 2022)

The generating function for the joint distribution of a quadruple (*des*, *ides*, *rmax*, *lmax*) of Euler-Stirling statistics on permutations equals (with $r = 1 - t$)

$$\mathcal{G}(t; x, u, v, 1, z) = \sum_{n=1}^{\infty} \frac{ztvr^{n-1}(1 + ux\, T_n)}{u(xr^{n-1} - 1)} \prod_{i=1}^{n-1} \frac{t(1 - z)r^{i-1} + r^i - 1}{u(r^i - 1)}$$

$$\times \left(1 - \sum_{n=1}^{\infty} \frac{t(z - 1)r^{n-1}}{u(r^n - 1)} \prod_{i=1}^{n-1} \frac{t(1 - z)r^{i-1} + r^i - 1}{u(r^i - 1)}\right)^{-1},$$

where $r = 1 - t$ and $T_n = r^{n-1}\mathcal{G}(t; xr^{n-1}, u, v, 1, 1)$ (see Thm 2).

Remark: $x^{-1}\mathcal{G}(tx; x^{-1}, u, v, 1, z)$ is the generating function of permutations with respect to the quadruple (des,iasc,rmax,lmax) or equivalently (iasc,des,lmax,lmin). Here asc$(\pi) = n - 1-$des(π) and iasc(π) =asc(π^{-1}).

Symmetric refined generating functions

Theorem 4 (J. 2022)

The generating function of permutations with respect to the triple (des,lmax,lmin), or equivalently (ides,lmax,lmin) of Euler-Stirling statistics is

$$\sum_{n=1}^{\infty}\sum_{\pi\in S_n}\frac{u^{des(\pi)}z^{lmax(\pi)}q^{lmin(\pi)}t^n}{(1-u)^n}$$
$$=\sum_{n=0}^{\infty}\frac{qztu^n}{1-(n-q+1)t}\prod_{i=0}^{n}\frac{1-(i-q+1)t}{1-(i+z)t}.$$

This generalizes a classical identity for the Eulerian numbers when $q=z=1$. Let us note that all these generating functions formulas hold as formal power series in u, so one can not specialize $u=1$.

3. Bijections and multiple equidistributions

Bijections and multiple equidistributions

Our proofs of main results consist of three decompositions of inversion sequences, a new bijection and transformation formulas on basic hypergeometric series.

Baril and Vajnovszki (2017) established a bijection $\Theta : S_n \to I_n$ satisfying that for any $\pi \in S_n$,

$$(des, ides, lmin, lmax, rmax)\pi = (asc, dist, max, zero, rmin)\Theta(\pi).$$

$$asc(s) := |\{i \in [n-1] : s_i < s_{i+1}\}|,$$
$$dist(s) := |\{s_1, s_2, \ldots, s_n\}| - 1,$$
$$zero(s) := |\{i \in [n] : s_i = 0\}|,$$
$$max(s) := |\{i \in [n] : s_i = i - 1\}|,$$
$$rmin(s) := |\{s_i : s_i < s_j \text{ for all } j > i\}|.$$

Remark: Of these five Euler-Stirling statistics, the most difficult one to keep track of is dist, because whether an entry is distinct or not depends on the entire sequence.

Bijections and multiple equidistributions

Define the set-valued statistic of rmin:

$$Rmin(s) := \{s_i : s_i < s_j \text{ for all } j > i\}.$$

Proposition 1 (J. 2022)

The quadruple of statistics (asc, zero, max, Rmin) has the same distribution as the one (dist, zero, max, Rmin) on inversion sequences of a given length.

Proof sketch:

We construct inversion sequences in two different ways. One records the change of asc, while the other one keeps track of the change of dist. Both procedures preserve the set Rmin.

This implies that (asc, zero, max, rmin) and (dist, zero, max, rmin) have the same distribution over inversion sequences, i.e., $\mathcal{G}(t; u, 1, q, v, z) = \mathcal{G}(t; 1, u, q, v, z)$.

4. Transformations of basic hypergeometric series

Transformations of basic hypergeometric series

For indeterminates a and q (the latter is referred to as the base), and non-negative integer k, the basic shifted factorial (or q-shifted factorial) is defined as

$$(a;q)_k := \prod_{j=1}^{k}(1-aq^{j-1}), \text{ also for } k=\infty.$$

For brevity, we write

$$(a_1,\dots,a_m;q)_k := (a_1;q)_k \cdots (a_m;q)_k.$$

${}_\alpha\phi_\beta$ basic hypergeometric series:

$${}_\alpha\phi_\beta\left[\begin{matrix} a_1,\dots,a_\alpha \\ b_1,\dots,b_\beta \end{matrix};q,z\right] := \sum_{k=0}^{\infty}\frac{(a_1,\dots,a_\alpha;q)_k z^k}{(q,b_1,\dots,b_\beta;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{1+\beta-\alpha}$$

where lower parameters b_i's are assumed to be chosen such that no poles occur in the summands of the series.

Transformations of basic hypergeometric series

Theorem (J.-Schlosser, 2020): Let a, b, c, d, e, r be complex variables, j be a non-negative integer. Then, assuming that none of the denominator factors on both sides of the identity have vanishing constant term in r, we have the following transformation of convergent power series in a and r:

$$
{}_4\phi_3\left[\begin{matrix}(1-r)^j, 1-a, b, c\\ d, e, (1-r)^{j+1}(1-a)bc/de\end{matrix}; 1-r, 1-r\right]
$$

$$
= \frac{((1-r)e, (1-r)(1-a)bc/de; 1-r)_j}{((1-r)(1-a)/e, (1-r)bc/de; 1-r)_j}
$$

$$
\times {}_4\phi_3\left[\begin{matrix}(1-r)^j, 1-a, d/b, d/c\\ d, de/bc, (1-r)^{j+1}(1-a)/e\end{matrix}; 1-r, 1-r\right].
$$

Transformations of basic hypergeometric series

Applying this theorem, we can show that the generating functions below are symmetric in u and x:

$$\mathcal{G}(t;x,u,v,1,1) = \sum_{n=1}^{\infty} \frac{vtu^{n-1}}{(1-t)^n - x} \prod_{i=1}^{n-1} \frac{x-(1-t)^i - xvt}{x-(1-t)^i}$$

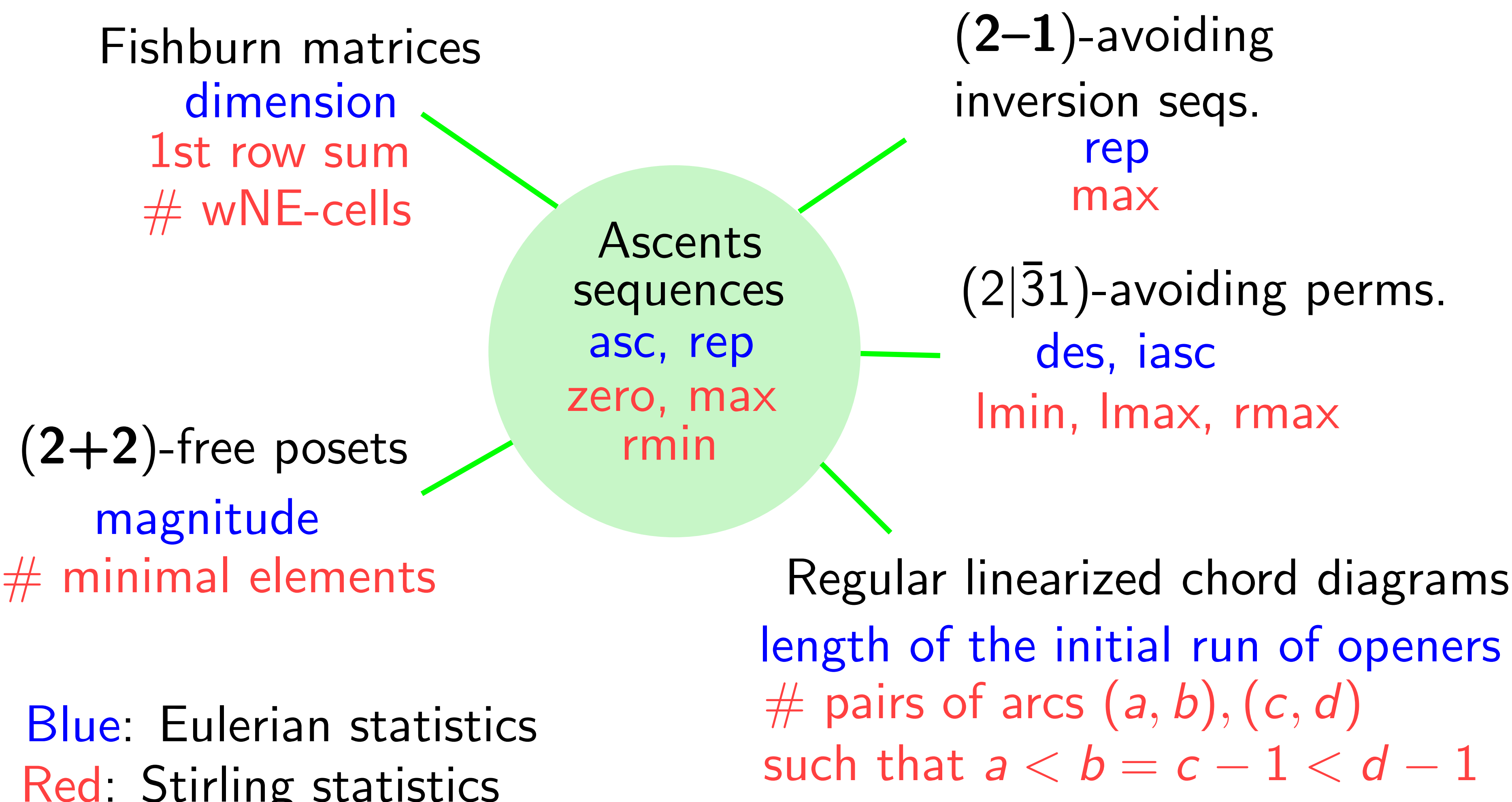
$$= \sum_{n=1}^{\infty} \frac{vt(1-t)^{n-1}}{u^n(x(1-t)^{n-1}-1)} \prod_{i=1}^{n} \frac{1-x(1-t)^{i-1}}{x(1-t)^{i-1}(vt-1)+1}$$

satisfy $\mathcal{G}(t;x,u,v,1,1) = \mathcal{G}(t;u,x,v,1,1)$.

Remark: J. Schlosser (2020) came up with this transformation formula of ${}_4\Phi_3$ on their way to count a subclass of permutations according to the Euler–Stirling statistics. To our surprise, it appears also in the refined enumerations of permutations.

5. Connections to other combinatorial structures

Our original motivation comes from recent work on members of Fishburn family (see Figure below):



Connections to other combinatorial structures

Question 1: Can we find the (bi)-symmetric properties in other bijective equivalent classes of permutations / inversion sequences?

Question 2: Can we prove the symmetric property in Thm 1 by manipulations on formal power series?

$$\mathcal{G}(t;x,u,v,q,1) = \frac{vt}{1-x}\sum_{n=1}^{\infty}\frac{qx-1+(1-q)r^{n-1}}{x-r^n}\prod_{i=1}^{n-1}\frac{u(x-r^i-xvt)((1-qt)r^{i-}}{(r^i-1)(x-r^i)}$$

$$\times\left(1-\frac{ut(q-1)}{x-1}\sum_{n=1}^{\infty}\frac{(x-1-xv)r^{n-1}+xv}{r^n-1}\right.$$

$$\left.\times\prod_{i=1}^{n-1}\frac{u(x-r^i-xvt)((1-qt)r^{i-1}-1)}{(r^i-1)(x-r^i)}\right)^{-1}.$$

Can we identity a new connection between classical transformation formulas of basic hypergeometric series and $\mathcal{G}(t;x,u,v,q,1)$?

Connections to other combinatorial structures

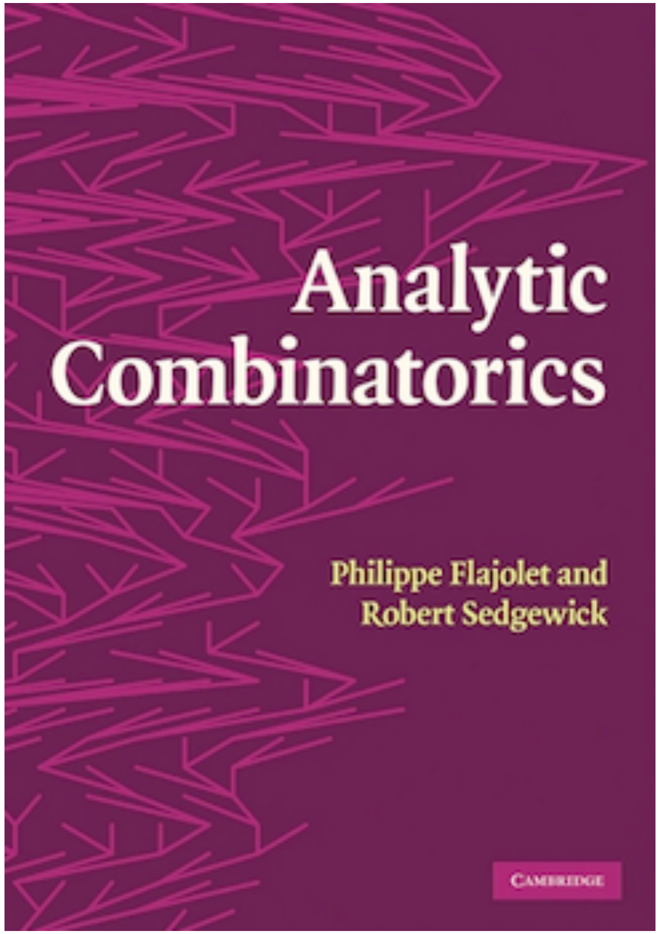
Question 3: What would the generating function look like if all Mahonian, Eulerian and Stirling statistics are included?

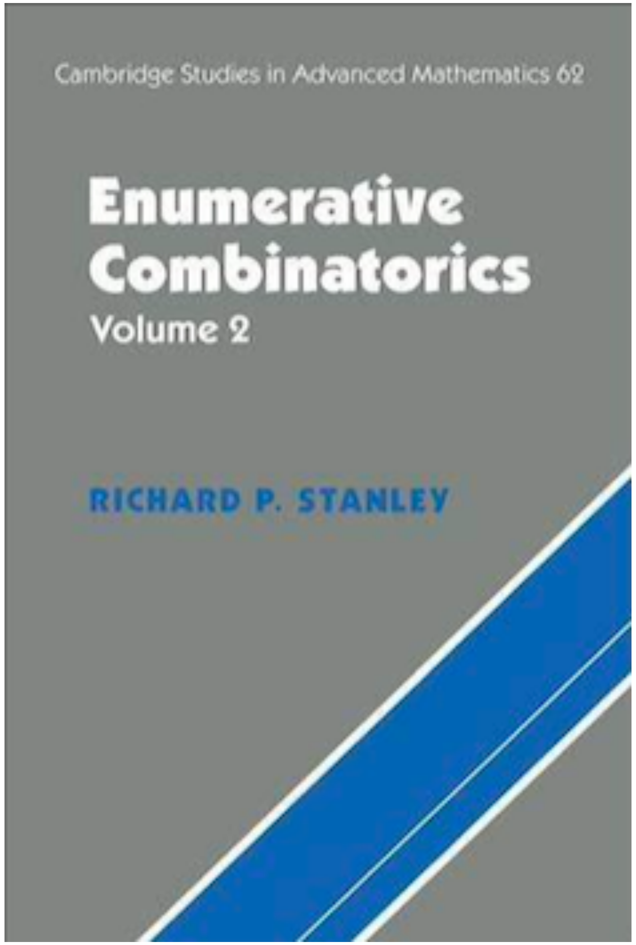
Question 4: The generating functions for the Fishburn structures with respect to the Euler–Stirling statistics have also a sum-of-finite-product form (like the one for permutations). Can we take a unified approach to treat class/subclass of permutations according to Euler–Stirling statistics?

Question 5: A conjecture proposed by J. and Schlosser (2020) remains unsolved: There is a bijection $\Theta : S_n \to S_n$ such that

$$(des, iasc, lmin, lmax, rmax)\pi = (des, iasc, rmax, lmax, lmin)\Theta(\pi).$$

Some books in this area





Thank you for your attentions!