Thickened strips and skew Schur functions

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Outline:

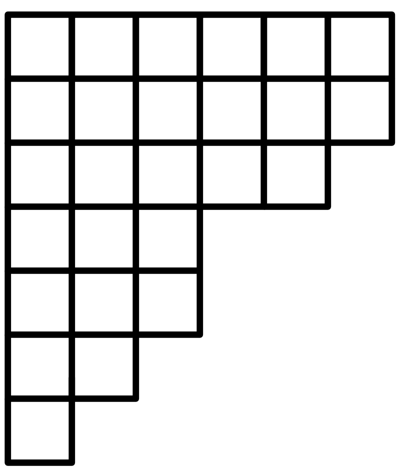
1. Young tableaux and Skew Schur functions
2. Border strips and thickened strips
3. Determinantal formulas of skew Schur functions
4. Non-intersecting lattice paths
5. Related work and future direction

1. Young tableaux and Skew Schur functions

Semistandard Young Tableaux (SSYT)

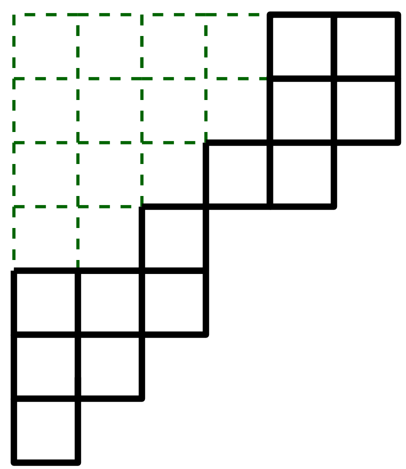
A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n is a weakly decreasing sequence of positive integers (i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$) such that the sum of λ_i's equals n. Each partition λ is represented as a Young diagram.

A skew partition λ/μ of n is the set-theoretic difference λ/μ of the Young diagrams, which consists of n boxes.



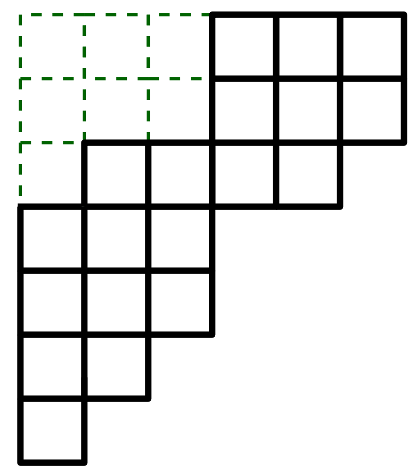
$\lambda = (6, 6, 5, 3, 3, 2, 1)$

edgewise disconnected



$\mu = (4, 4, 3, 2)$

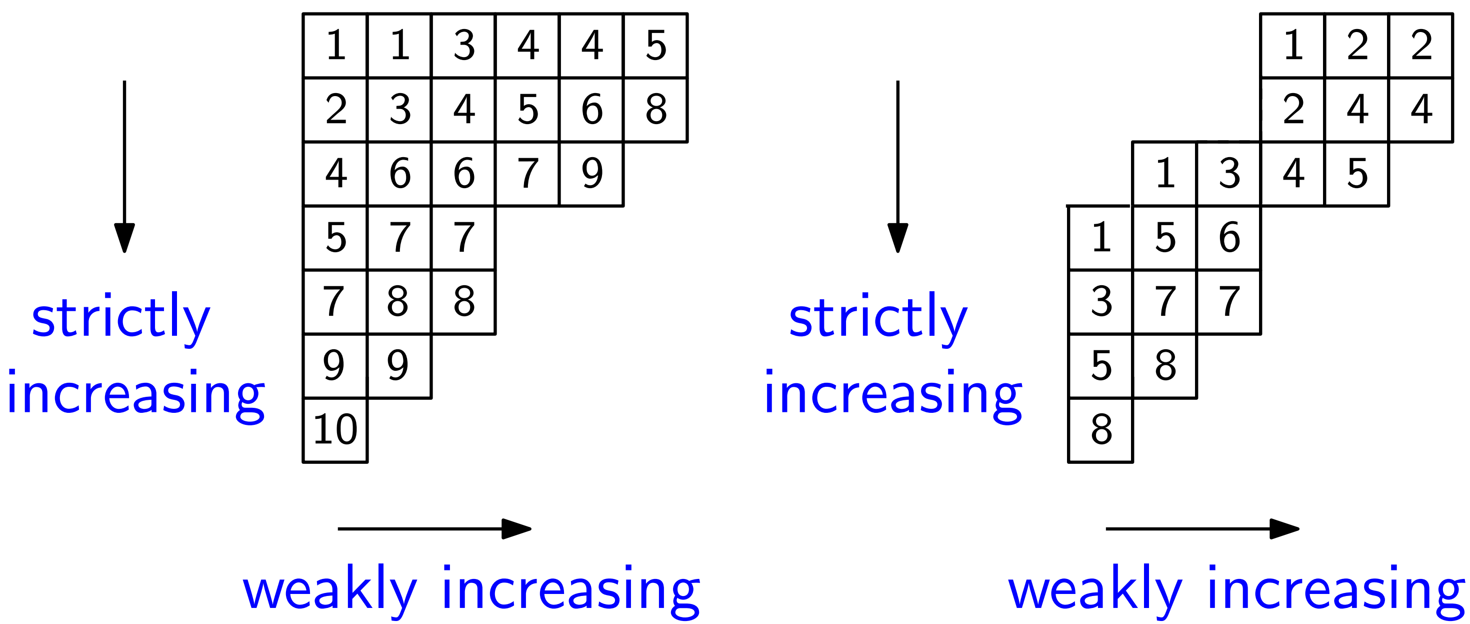
edgewise connected



$\mu = (3, 3, 1)$

Semistandard Young Tableaux (SSYT)

A Semistandard Young tableau (SSYT) of shape λ/μ is a Young diagram whose boxes have been filled with positive integers such that all entries along each row from left to right are weakly increasing, while all entries along each column from top to bottom are strictly increasing.

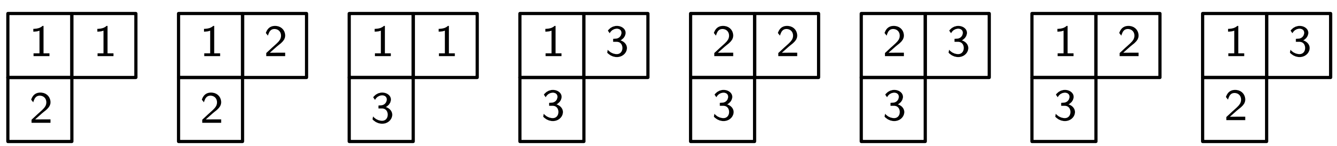


Semistandard Young Tableaux (SSYT)

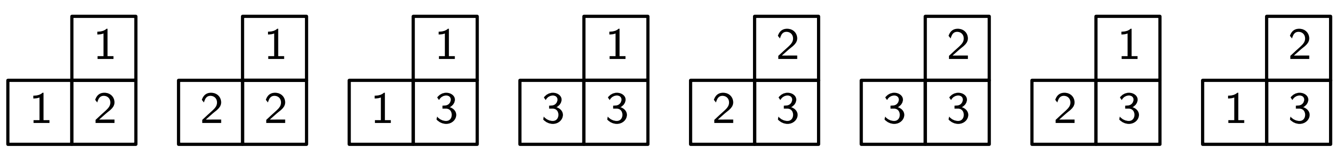
The skew Schur function $s_{\lambda/\mu}(x)$ of shape λ/μ in the infinitely many variables $x = (x_1, x_2, \ldots,)$ is the formal power series

$$s_{\lambda/\mu}(x) = \sum_T x^T,$$

summed over all SSYTs of shape λ/μ. In particular if $\mu = \emptyset$, we call $s_\lambda(x)$ the Schur function of shape λ.



$$s_{21}(x_1, x_2, x_3) = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + 2x_1 x_2 x_3.$$

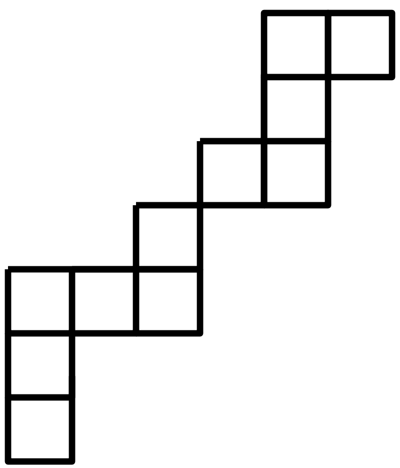


$$s_{(22)/(1)}(x_1, x_2, x_3) = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + 2x_1 x_2 x_3.$$

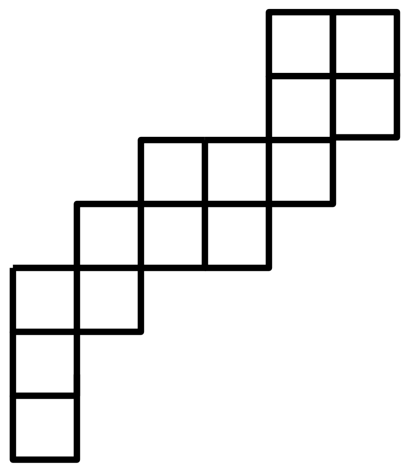
2. Border strips and thickened strips

Border strips and thickened border strips

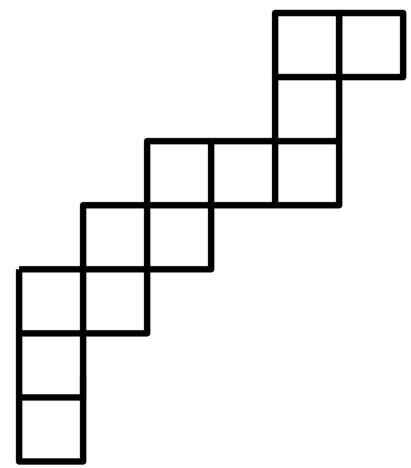
A border strip is an edgewise connected skew shape without 2×2 blocks of boxes. A thickened border strip is an edgewise connected skew shape without 2×3 or 3×2 blocks of boxes.



edgewise disconnected



thickened border strip



border strip

3. Determinantal formulas of skew Schur functions

Skew Schur function determinants

Jacobi-Trudi determinant and its dual $s_{\lambda/\mu}(X)$

Giambelli determinant $s_\lambda(X)$

Lascoux and Pragacz determinant $s_{\lambda/\mu}(X)$

unified by

Hamel and Goulden (1995) determinant $s_{\lambda/\mu}(X)$

Chen, Yan and Yang (2005) introduced cutting strips

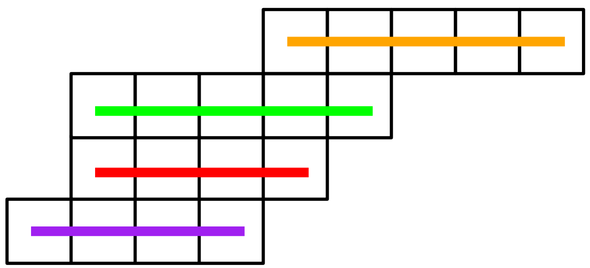
allows common boxes

J., 2018, a determinant $p_{1^r}(X)s_{\lambda/\mu}(X)$
(outside nested decompositions)

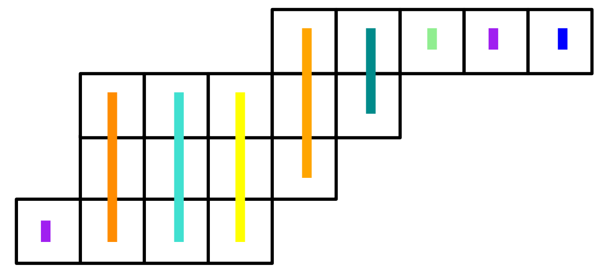
to count m-strip tableaux

by Baryshnikov and Romik (2010)

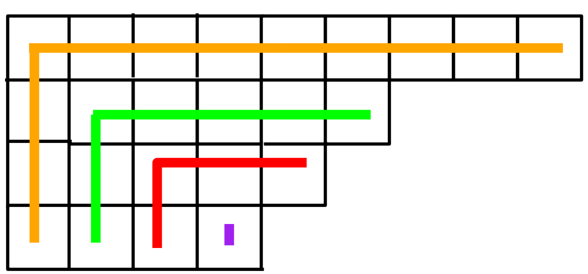
Skew Schur function determinants



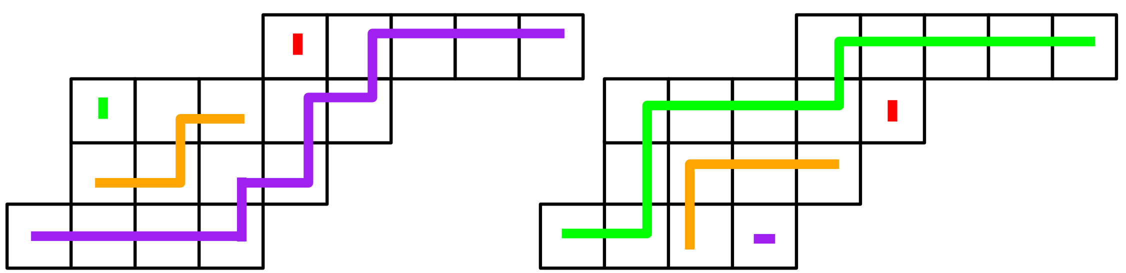
Jacobi-Trudi determinant



Dual Jacobi-Trudi determinant



Giambelli determinant
(only standard shape)



Lascoux and Pragacz determinant
choose the outer/inner strips

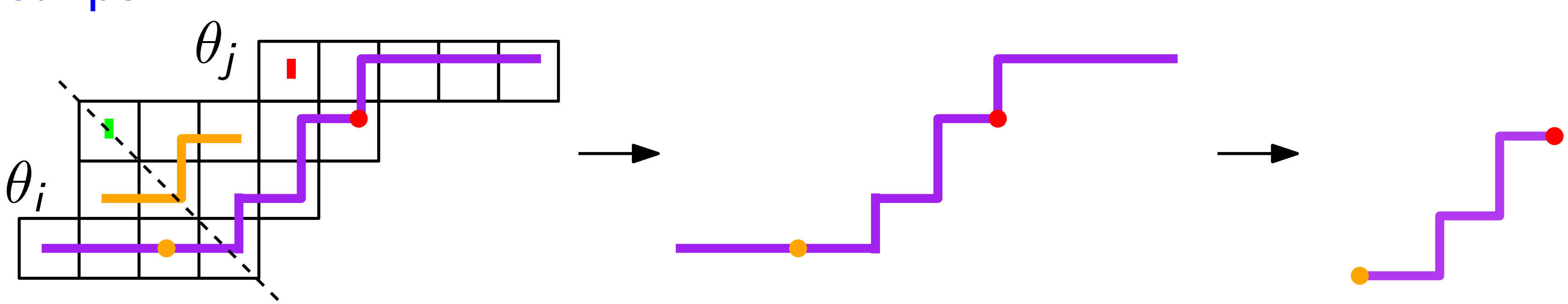
Skew Schur function determinants

Theorem (Hamel and Goulden, 1995, Eur. J. Comb.). If the skew diagram of λ/μ is edgewise connected. Then, for any outside decomposition $\theta = (\theta_1, \theta_2, \ldots, \theta_k)$ of skew shape λ/μ, we have

$$s_{\lambda/\mu}(x) = \det[s_{\theta_i \# \theta_j}(x)]_{i,j=1}^{k},$$

where $s_{\emptyset}(x) = 1$ and $s_{\theta_i \# \theta_j}(x) = 0$ if $\theta_i \# \theta_j$ is undefined.

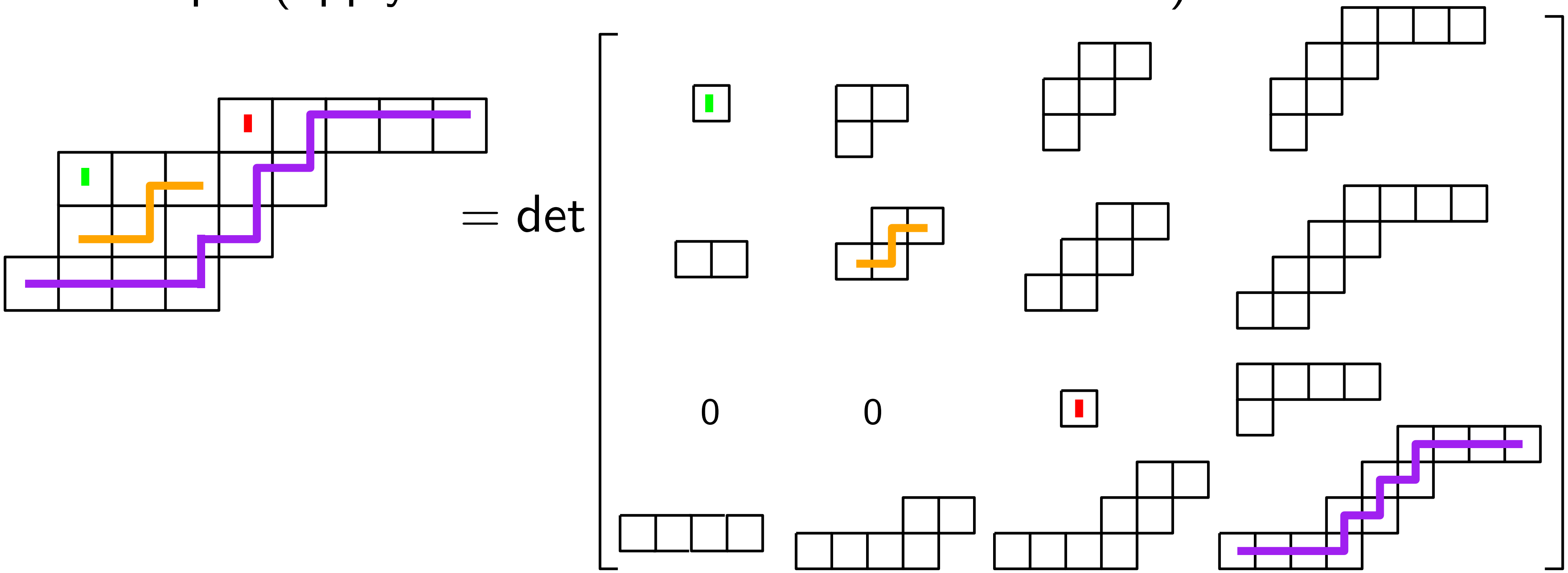
Chen, Yan and Yang (2005, J. Algebr. Comb.) largely simplified the definition of $\theta_i \# \theta_j$ by introducing the concept of cutting strips.



collecting border strips along each diagonal | the cutting strip | $\theta_i \# \theta_j$

Skew Schur function determinants

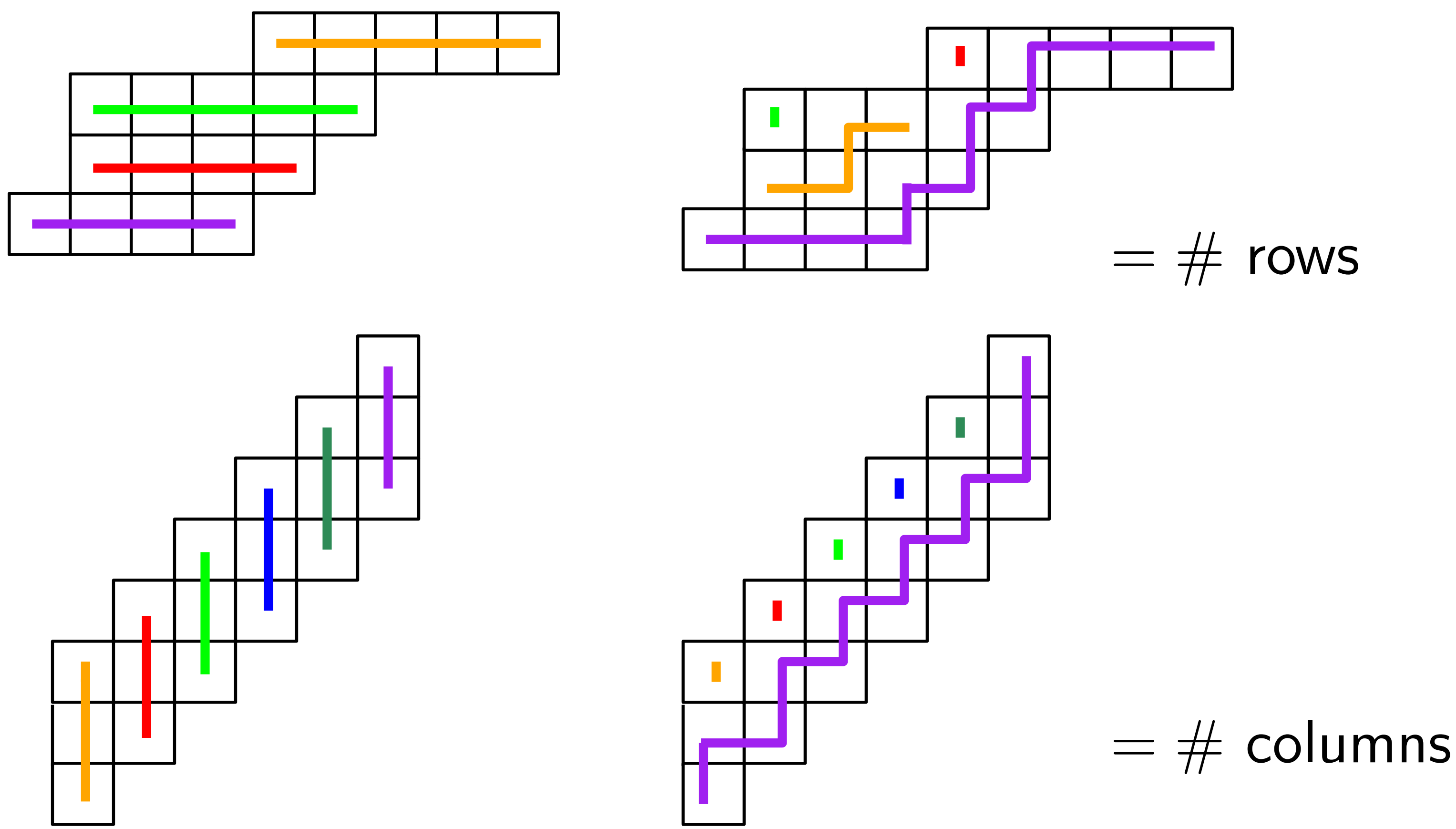
Example (apply the Hamel and Goulden Theorem):



Here a skew shape D refers to the corresponding skew Schur function $s_D(x)$. By the Hamel–Goulden Theorem, we always choose the minimal number of border strips in an outside decomposition, so that the order of the determinant is minimized.

Skew Schur function determinants

However, for some skew shape, the minimal number of border strips equals the number of rows or columns. The order of the determinant can not be reduced by the Hamel–Goulden Theorem.



Skew Schur function determinants

Theorem (Jin 2018, Eur. J. Comb.). If the skew diagram of λ/μ is edgewise connected. Then, for any outside nested decomposition $\Theta = (\Theta_1, \Theta_2, \ldots, \Theta_k)$ of skew shape λ/μ, we set $r = \sum_{i=1}^{k} |\Theta_i| - |\lambda/\mu|$ and we have

$$p_{1^r}(x) s_{\lambda/\mu}(x) = \det[s_{\Theta_i \# \Theta_j}(x)]_{i,j=1}^{k}, \text{ where } p_{1^r}(x) = (\sum_{i=1}^{\infty} x_i)^r,$$

$s_{\emptyset}(x) = 1$ and $s_{\Theta_i \# \Theta_j}(x) = 0$ if $\Theta_i \# \Theta_j$ is undefined.

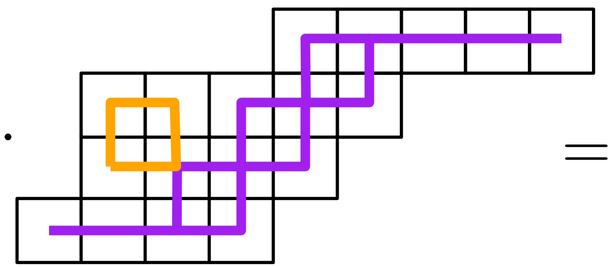
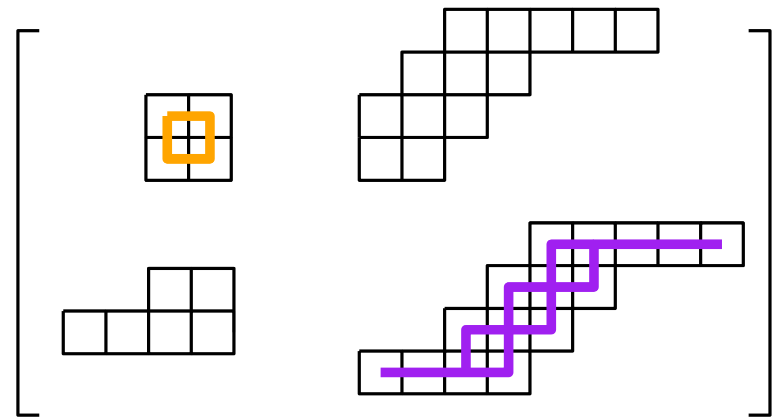
Remark 1: All Θ_i's are thickened border strips. If all Θ_i's are disjoint strips, then we retrieve the HG Theorem.

Remark 2: Through outside nested decomposition, we can further reduce the order of the skew Schur determinants.

Remark 3: We allow thickened strips to have common boxes.

Skew Schur function determinants

Example (apply the J. Theorem):

$(\sum_{i=1}^{\infty} x_i)\cdot$  $= \det$ 

Here a skew shape D refers to the corresponding skew Schur function $s_D(x)$. By the J. Theorem, we obtain a simplified determinant of order 2. Each thickened border strip in contrast with border strips may contain 2×2 blocks of boxes. Subsequently we will see that each thickened border strip is not difficult to count.

Skew Schur function determinants

A standard Young tableau (SYT) of shape λ/μ is SSYT filled with integers from 1 to $|\lambda/\mu|$. In other words, all entries along each row/column are strictly increasing from left/top to right/bottom. Let $f^{\lambda/\mu}$ be the number of SYTs of shape λ/μ. Then,

> Corollary (J. 2018). If the skew diagram of λ/μ is edgewise connected. Then, for any outside nested decomposition $\Theta = (\Theta_1, \Theta_2, \ldots, \Theta_k)$ of skew shape λ/μ, we have
>
> $$f^{\lambda/\mu} = |\lambda/\mu|! \det[(a_{i,j}!)^{-1} f^{\Theta_i \# \Theta_j}]_{i,j=1}^{k}, \text{ where } a_{i,j} = |\Theta_i \# \Theta_j|,$$
>
> $f^{\emptyset} = 1$ and $f^{\Theta_i \# \Theta_j} = 0$ if $\Theta_i \# \Theta_j$ is undefined.

Remark: Interestingly, the number r of common boxes of Θ_i's plays no role in the formula of $f^{\lambda/\mu}$. Again this corollary enables us to reduce the order of the determinant.

4. Non-intersecting lattice paths

Non-intersecting lattice paths

Proof techniques of Hamel–Goulden Theorem:

Semistandard Young Tableaux (SSYTs)

↓ 1-1

Non-intersecting lattice paths

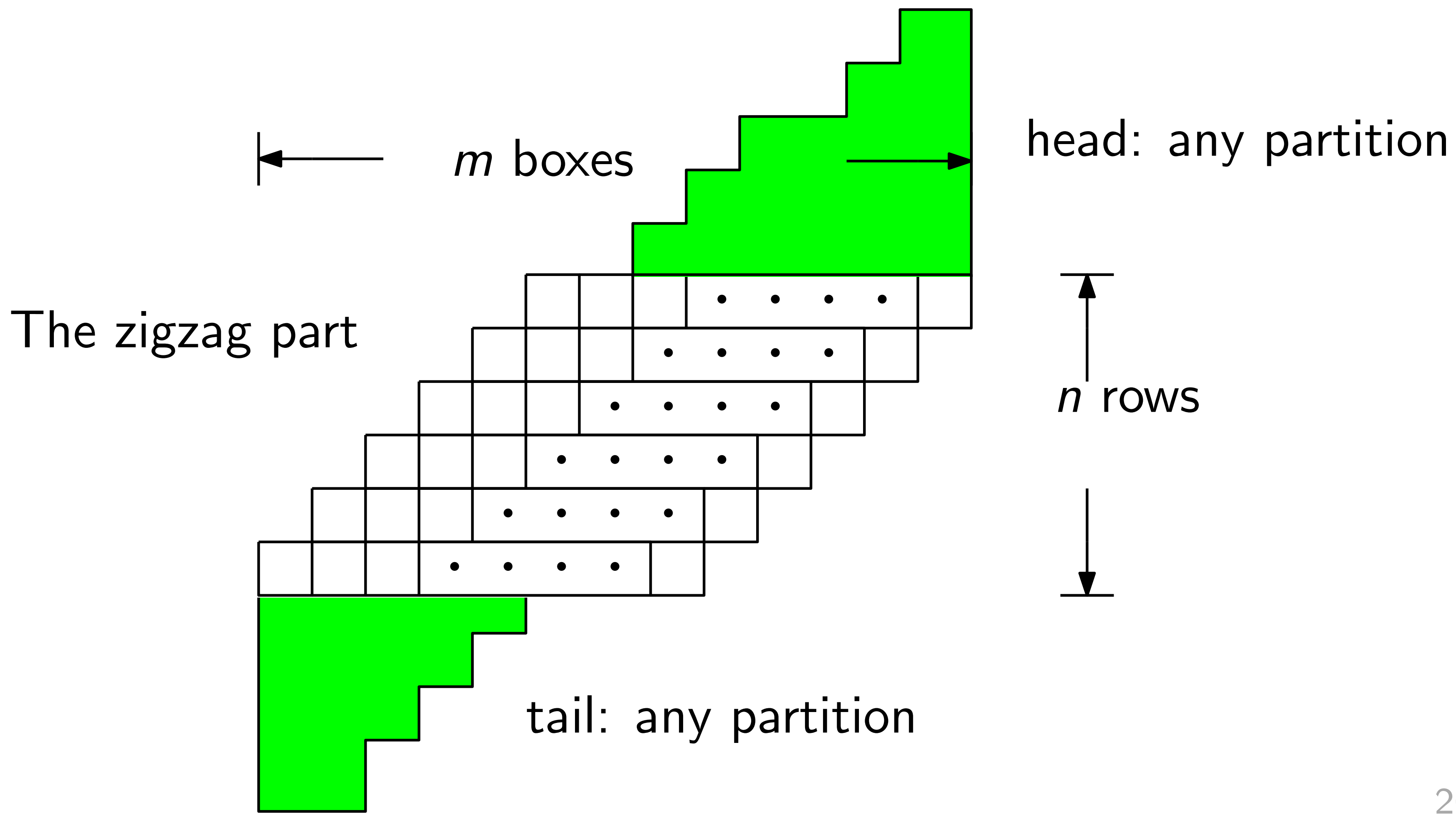
↓ by Stembridge's Theorem
developed by Lindström-Gessel-Viennot's approach

Construct an involution on lattice paths

Our innovative part: We instead allow the lattice paths to be intersecting and control the intersecting parts within certain range so that they still cancel each other under the involution. In other words, we took non-intersecting approach to resolve certain intersecting lattice paths problem.

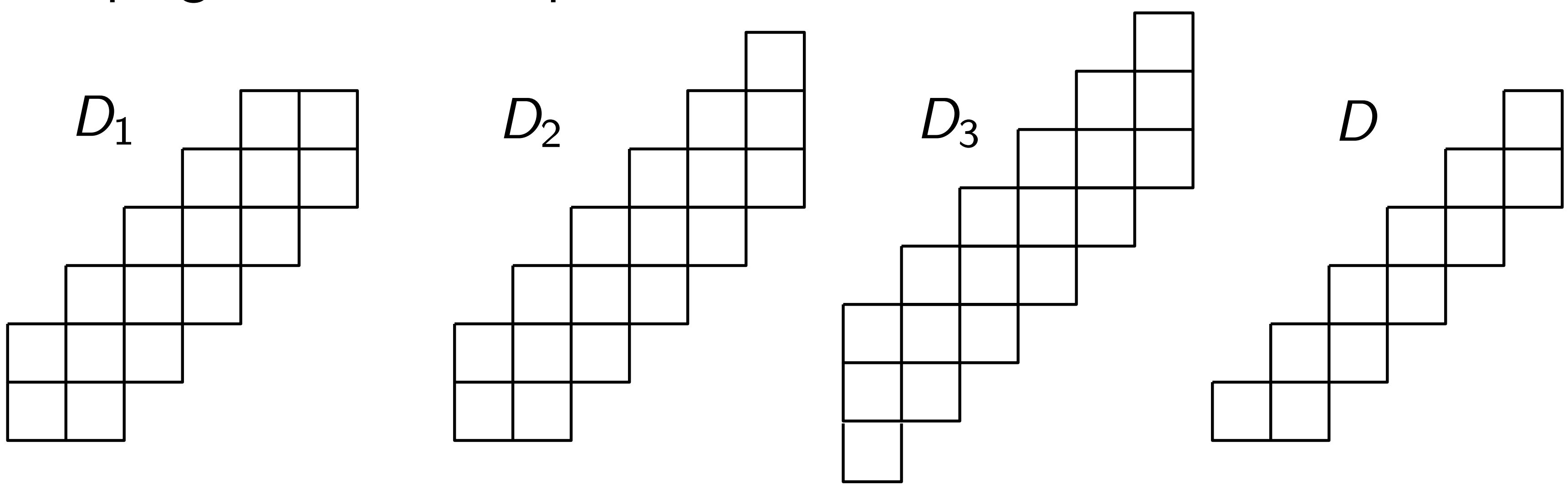
Applications to zigzag-strips

The zigzag strips were introduced by Baryshnikov and Romik (2007, Isr. J. Math.) to extend the transfer operator approach due to Elkies.



Applications to zigzag-strips

For instance, let D_1, D_2, D_3 be all zigzag skew shapes for the case $m = 3$, then we derive the number of SYTs of shape D_i by developing a new decomposition of D_i.



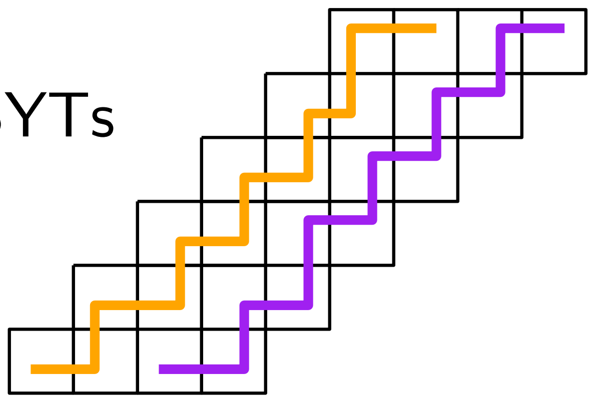
It is well-known that $f^D = E_{2n-1}$, the Euler number, which counts the number of alternating permutations of $[2n-1]$.

$$f^{D_1} = \frac{(3n-2)!E_{2n-1}}{(2n-1)!2^{2n-2}}, \quad \frac{f^{D_2}}{f^{D_1}} = \frac{3n-1}{2}, \frac{f^{D_3}}{f^{D_2}} = \frac{(3n)(2^{2n-1}-1)}{2^{2n}-1}.$$

Applications to zigzag-strips

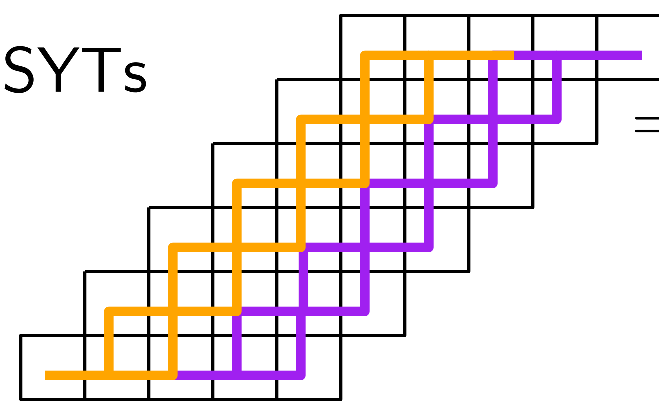
For the case $m = 4$, we can directly apply the Hamel–Goulden Theorem, while for $m = 5$, we have to use J. Theorem.

$\#$ SYTs



$$= (4n)!\det\begin{bmatrix} E_{2n} & E_{2n+2} \\ E_{2n-2} & E_{2n} \end{bmatrix}$$

$\#$ SYTs



$$= (5n)!\det\begin{bmatrix} f^{D_n} & f^{D_{n+1}} \\ f^{D_{n-1}} & f^{D_n} \end{bmatrix}$$ where

$$f^{D_n} = \frac{(3n)!(2^{2n-1}-1)E_{2n-1}}{(2n-1)!2^{2n-1}(2^{2n}-1)}.$$

5. Related work and future direction

Related work and future direction

Theorem (J. 2018). If the skew diagram of λ/μ is edgewise connected. Then, for any outside nested decomposition $\Theta = (\Theta_1, \Theta_2, \ldots, \Theta_k)$ of skew shape λ/μ, we set $r = \sum_{i=1}^{k} |\Theta_i| - |\lambda/\mu|$ and we have

$$p_{1^r}(x) s_{\lambda/\mu}(x) = \det[s_{\Theta_i \# \Theta_j}(x)]_{i,j=1}^{k}, \text{ where } p_{1^r}(x) = (\sum_{i=1}^{\infty} x_i)^r,$$

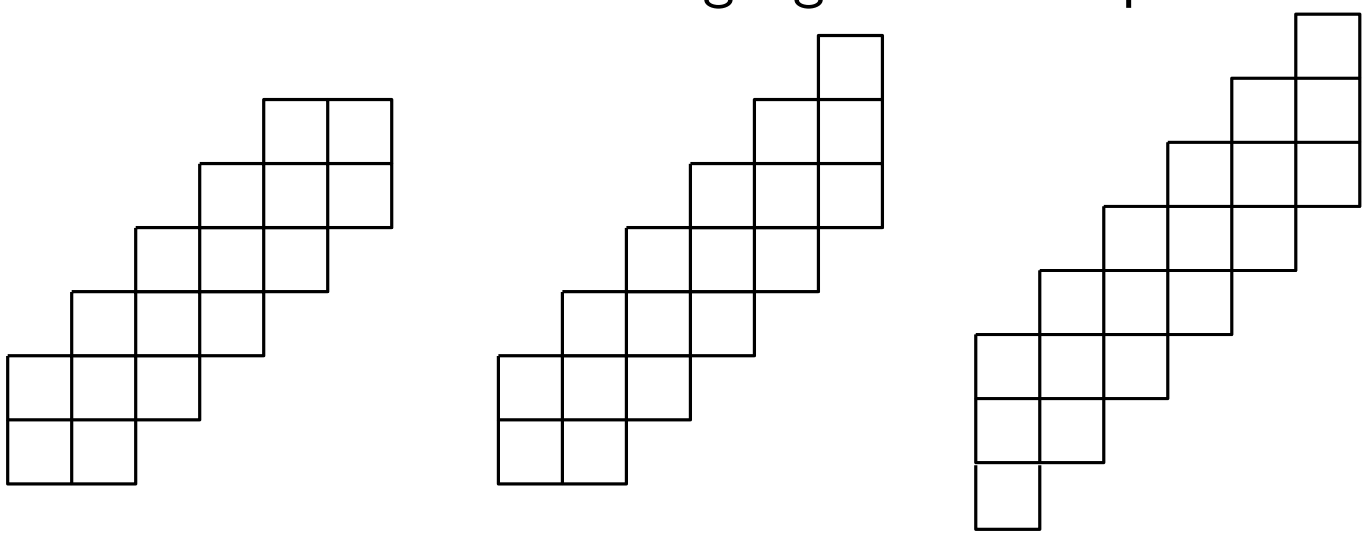
$s_{\emptyset}(x) = 1$ and $s_{\Theta_i \# \Theta_j}(x) = 0$ if $\Theta_i \# \Theta_j$ is undefined.

1. Our proof is purely combinatorial. Can we find an algebraic proof of this theorem? For the case $\mu = \emptyset$, Kim and Yoo (2021) provided such a proof by applying the Bazin identity.

2. Can we extend this result to other generalizations of the skew Schur functions and Schur Q-functions?

Related work and future direction

3. There are many refinements on the enumeration of alternating permutations. Can we carry out a parallel study on the enumeration of thickened zigzag border strips?



4. Non-intersecting lattice paths have wide applications in Enumerative Combinatorics. Can we apply the idea to other combinatorial structures, namely to treat certain intersecting paths?

Thank you!