# Asymptotics and Statistics on Fishburn matrices

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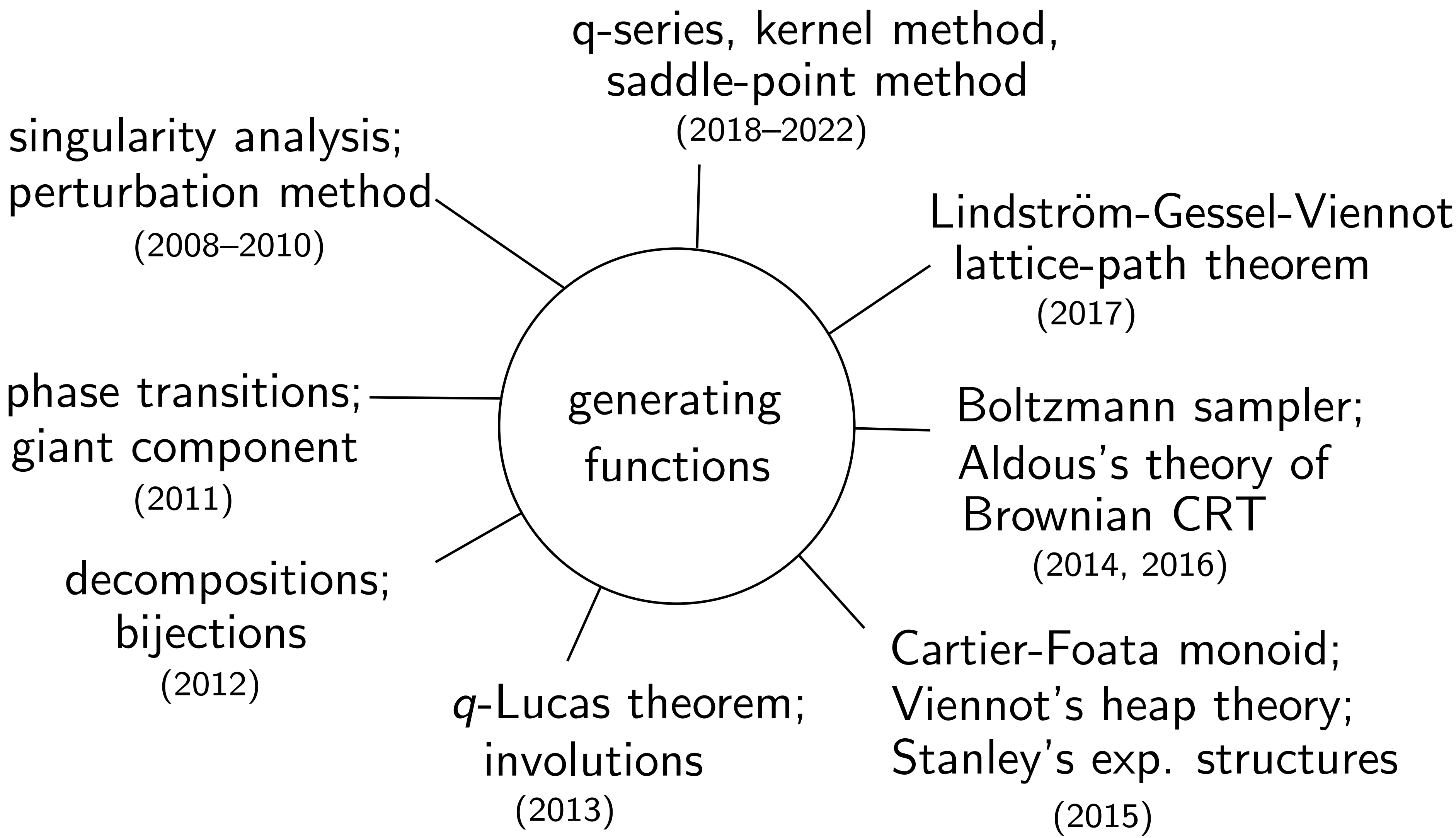
# An overview of my previous research



Outline:

1. Combinatorial background of the Fishburn family
2. Generating functions and $q$-series
3. Transformations of basic hypergeometric series
4. Asymptotics and statistics on Fishburn structures
5. Concluding remarks

## 1. Combinatorial background of the Fishburn family

# Fishburn matrices

Fishburn matrices are non-negative, upper-triangular square matrices with at least one positive entry in each row and column.

E.g. there are 15 Fishburn matrices of size (the sum of entries) 4:

$$\begin{pmatrix} 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The number of Fishburn matrices of a given size are known as Fishburn numbers (A022493 of the OEIS). Their first few values are $(1, 2, 5, 15, 53, 217, 1014, 5335, 31240, 201608, \ldots)$

# Fishburn numbers

Fishburn numbers are the coefficients of the formal power series

$$\sum_{m=0}^{\infty}\prod_{i=1}^{m}(1-(1-z)^i) = 1 + z + 2z^2 + 5z^3 + 15z^4 + 53z^5 + \cdots,$$

which count the Fishburn matrices of a fixed dimension. This generating function was derived by Zagier (2001). Subsequently Andrews and Jelínek (2013) found an equivalent form:

$$\sum_{m=0}^{\infty}\prod_{i=1}^{m}(1-(1-z)^i) = \sum_{k=0}^{\infty}\frac{1}{(1-z)^{k+1}}\prod_{i=1}^{k}\left(1-\left(\frac{1}{1-z}\right)^i\right)^2$$

by applying the Rogers–Fine identity:

$$\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(bq;q)_n}t^n = \sum_{n=0}^{\infty}\frac{(aq;q)_n(\frac{atq}{b};q)_n b^n t^n q^{n^2}(1-atq^{2n+1})}{(bq;q)_n(t;q)_{n+1}}$$

holds when $|q|<1$, $|t|<1$ and $b \neq q^k$ for $k<0$.

# Fishburn numbers

The study of Fishburn numbers and their generalizations has remarkably led to many interesting results, including for instance

- Congruences (Garvan 2015, Andrews–Sellers 2016, Bijaoui–Boden–Myers–Osburn–Rushworth–Tronsgard–Zhou 2020),
- Asymptotic formulas (Zagier 2001, Jelínek 2012, Bringmann–Li–Rhoades 2014, Hwang–J. 2019),
- $q$-series (Andrews–Jelínek 2013, J.–Schlosser 2020),
- A variety of bijections (Bousquet-Mélou–Claesson–Dukes–Kitaev 2010, Claesson–Linusson 2011, Dukes–Parviainen 2010, Levande 2013, Fu–J.–Lin–Yan–Zhou 2019, Dukes–McNamara 2019, Auli–Elizalde 2020),
- Generating functions (Bousquet-Mélou–Claesson–Dukes–Kitaev 2010, Dukes–Kitaev–Remmel–Steingrímsson 2011, Jelínek 2012, Zagier 2001).

# Statistics on members of the Fishburn family

Ascent sequences
asc, rep
zero, max
rmin

$(\mathbf{2{-}1})$-avoiding inversion seqs.
rep
max

Fishburn matrices
dimension
1st row sum
$\#$ wNE-cells

$(2|\bar{3}1)$-avoiding perms.
des, iasc
lmin, lmax, rmax

$(\mathbf{2+2})$-free posets
magnitude
$\#$ minimal elements

Regular linearized chord diagrams
length of the initial run of openers
$\#$ pairs of arcs $(a, b), (c, d)$ such that $a < b = c - 1 < d - 1$

Blue: Eulerian statistics
Red: Stirling statistics

# 2. Generating functions and $q$-series

# Refined generating functions

Let $\mathcal{F}(z, v)$ be the generating function of Fishburn matrices with respect to size (variable $z$) and dimension (variable $v$):

$$
\begin{aligned}
F(z, v) &:= \sum_{n=1}^{\infty} z^n \sum_{A \in \mathcal{F}_n} v^{dim(A)}, \\
&= \sum_{k=1}^{\infty} \frac{v\,(1-z)^k}{1 - v + v(1-z)^k} \prod_{j=1}^{k} (1 - (1-z)^j), \\
&= \sum_{k=0}^{\infty} v^{k+1} \prod_{j=1}^{k+1} \frac{1 - (1-z)^j}{v + (1-v)(1-z)^j}.
\end{aligned}
$$

The first one was derived by Jelínek (2012) via the Fishburn matrices, and the second one was found by Fu-J.-Lin-Yan-Zhou (2019) by a new decomposition of ascent sequences. Subsequently, J.-Schlosser (2022) used the Sears transformation to establish more equivalent forms.

# Fishburn matrices and two variations

| | Generating functions | Definitions |
|---|---|---|
| row-Fishburn matrices | $\sum_{m=1}^{\infty}\prod_{i=1}^{m}((1-z)^{-i}-1)$ | non-negative; upper-triangular; each row has one positive entry. |
| Fishburn matrices | $\sum_{m=1}^{\infty}\prod_{i=1}^{m}(1-(1-z)^{i})$ | $+$ each column has one positive entry. |
| self-dual Fishburn matrices | $\sum_{m=0}^{\infty}(1-z)^{-m-1}\prod_{i=1}^{m}((1-z^2)^{-i}-1)$ | $+$ persymmetric |

Ref. Jelínek, Counting general and self-dual interval orders, 2012.
Zagier, Vassiliev invariants and a strange identity related to the Dedekind eta-function, 2001.

# 3. Transformations of basic hypergeometric series

# Transformations of basic hypergeometric series

For indeterminates $a$ and $q$ (the latter is referred to as the base), and non-negative integer $k$, the basic shifted factorial (or $q$-shifted factorial) is defined as

$$(a;q)_k := \prod_{j=1}^{k}(1-aq^{j-1}), \quad \text{also for } k=\infty.$$

For brevity, we write

$$(a_1,\ldots,a_m;q)_k := (a_1;q)_k \cdots (a_m;q)_k.$$

The Rogers-Fine identity:

$$\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(bq;q)_n}t^n = \sum_{n=0}^{\infty}\frac{(aq;q)_n(\frac{atq}{b};q)_n b^n t^n q^{n^2}(1-atq^{2n+1})}{(bq;q)_n(t;q)_{n+1}}$$

holds when $|q|<1$, $|t|<1$ and $b \neq q^k$ for $k<0$.

# A generalized Rogers–Fine identity

A generalized Rogers–Fine identity due to Andrews–Jelínek (2013): For any $r$ and $\gamma$, they proved the following identity of formal power series in $x$ and $y$:

$$\sum_{n=0}^{\infty} \frac{(\gamma(r(1-x))^{-1}; 1-x)_n((1-y)^{-1}; (1-x)^{-1})_n}{(\gamma; 1-x)_n} r^n$$

$$= \sum_{n=0}^{\infty} (1-y)(1-x)^n \frac{(1-y; 1-x)_n(r(1-x); 1-x)_n}{(\gamma; 1-x)_n},$$

where $(a; q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1})$.

This is a generalized Rogers–Fine identity. Andrews–Jelínek also asked for a combinatorial interpretation of this identity. We provide one in terms of Fishburn matrices for a special case of this identity.

# A generalized Rogers–Fine identity

Substituting $r = 1$, $\gamma = (v-1)v^{-1}(1-z)^{-1}$, $x = y = 1-(1-z)^{-1}$ shows that

$$\sum_{k=0}^{\infty} \frac{(1-z)^k}{1-v+v(1-z)^k} \prod_{j=1}^{k} (1-(1-z)^j)$$

$$= \sum_{k=0}^{\infty} \frac{1}{(1-z)^{k+1}} \prod_{j=1}^{k} \frac{((1-z)^{-j}-1)^2}{1-(v-1)v^{-1}(1-z)^{-j}}.$$

Recall that Jelínek (2012) found the generating function of Fishburn matrices with respect to dimension and size

$$F(z,v) := \sum_{n=1}^{\infty} z^n \sum_{A \in \mathcal{F}_n} v^{dim(A)}$$

$$= \sum_{k=1}^{\infty} \frac{v\,(1-z)^k}{1-v+v(1-z)^k} \prod_{j=1}^{k} (1-(1-z)^j).$$

# A generalized Rogers–Fine identity

This leads to an equivalent form of $F(z,v)$:

$$\begin{aligned} F(z,v) &= -v + \sum_{k=0}^{\infty} \frac{v}{(1-z)^{k+1}} \prod_{j=1}^{k} \frac{v((1-z)^{-j}-1)^2}{1-(v-1)((1-z)^{-j}-1)}, \\ &= \sum_{k=1}^{\infty} \frac{v\,(1-z)^k}{1-v+v(1-z)^k} \prod_{j=1}^{k} (1-(1-z)^j), \\ &= \sum_{k=0}^{\infty} v^{k+1} \prod_{j=1}^{k+1} \frac{1-(1-z)^j}{v+(1-v)(1-z)^j}. \end{aligned}$$

The first form of $F(z,v)$ is more suitable (than the other two below) for the saddle-point approach. While the Taylor expansion of inner product still contains, in general, negative coefficients, it plays asymptotically only a perturbative role when $v$ is close to 1.

# 4. Asymptotics and statistics on Fishburn structures

# Asymptotics of Fishburn numbers

Let $f_n$ be the $n$-th Fishburn number, that is,

$$f_n = [z^n] \sum_{m=1}^{\infty} \prod_{i=1}^{m} (1 - (1 - z)^i).$$

Theorem (Zagier, 2001)

$$f_n = n! \left( \frac{6}{\pi^2} \right)^n \sqrt{n} \left( \frac{12\sqrt{3}}{\pi^{5/2}} e^{\pi^2/12} + O(n^{-1}) \right).$$

Remark: Zagier first guessed the formula numerically, then he proved an identity via the modular-form approach, by which the estimation of $f_n$ follows immediately. For more details, we refer to the paper:

Ref. Zagier, Vassiliev invariants and a strange identity related to the Dedekind eta-function. Topology, 2001.

# Asymptotics of Fishburn numbers

How to guess the formula numerically (see Jelínek's talk in PP 2017):

1. Compute $f_n$ for large $n = N$ (as large as possible).
2. Make the Ansatz $f_n \approx c^n\, n!\, n^{\alpha}$ for some constant $c$.
3. Define $r_n = f_{n+1}/(nf_n)$, then $\lim_{n\to\infty} r_n = c$. Wanted $c$.
4. For a sequence $(a_n)_{n\in N}$, let $\Delta(a_n) = a_{n+1} - a_n$.
5. Observe for fixed integer $d \neq 0$, $\Delta(n^d) = dn^{d-1} + O(n^{d-2})$.
6. Suppose $r_n = c + \alpha_1 n^{-1} + \alpha_2 n^{-2} + \cdots$ for constants $\alpha_i$.
7. Then $nr_n = cn + \alpha_1 + \alpha_2 n^{-1}$ and $\Delta(nr_n) = c + O(n^{-2})$.
8. For fixed positive integer $k$, $\Delta^{(k)}(n^k r_n k!^{-1}) = c + O(n^{-k-1})$.
9. Set $k = 1000$ and define $t_n = \Delta^{(k)}(n^k r_n k!^{-1})$. Then $|t_{1000} - 6/\pi^2| < 10^{-180}$, suggesting that $t_n \to 6/\pi^2$ and also $r_n \to 6/\pi^2$.

# Asymptotics of row-Fishburn numbers

Let $g_n$ be the number of row-Fishburn matrices such that the sum of all entries is $n$. By using the numerical techniques from Zagier, Jelínek (2012) conjectured that

$$g_n = n! \left(\frac{12}{\pi^2}\right)^n \left(\frac{6\sqrt{2}}{\pi^2} e^{\pi^2/24} + O(n^{-1})\right).$$

This conjecture was affirmed by Bringmann, Li and Rhoades (2014).

$$\sum_{n\geq 1}^{\infty} g_n z^n = \sum_{n=0}^{\infty} \prod_{i=0}^{n} \left(\frac{1}{(1-z)^{i+1}} - 1\right).$$

$$(g_n)_{n=1}^{8} = (1, 3, 12, 61, 380, 2815, 24213, 237348 \ldots)$$

Ref. Bringmann, Y. Li and R.C. Rhoades, Asymptotics for the number of row-Fishburn matrices, Eur. J. Comb., 2014.

# Fishburn matrices and two variations

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| Fishburn matrices | $\displaystyle\sum_{m=1}^{\infty}\prod_{i=1}^{m}(1-(1-z)^{i})$ | + each column has one positive entry. |
| self-dual Fishburn matrices | $\displaystyle\sum_{m=0}^{\infty}(1-z)^{-m-1}\prod_{i=1}^{m}((1-z^2)^{-i}-1)$ | + persymmetric |

Ref. Jelínek, Counting general and self-dual interval orders, 2012.
Zagier, Vassiliev invariants and a strange identity related to the Dedekind eta-function, 2001.

# Asymptotics of self-dual Fishburn numbers

Let $r_n$ be the number of self-dual Fishburn matrices such that the sum of all entries is $n$. By using the numerical techniques from Zagier, Jelínek conjectured that

$$r_n = (\gamma + O(n^{-1/2}))\sqrt{n}\left(\frac{6n}{\pi^2 e}\right)^{n/2} 2^{\sqrt{6n}/\pi},$$

where $\gamma \approx 1.361951039$.

Jelínek presented the generating function of $r_n$:

$$\sum_{n\geq 1}^{\infty} r_n t^n = \sum_{n=0}^{\infty} \frac{1}{(1-x)^{n+1}} \prod_{i=1}^{n} \left(\frac{1}{(1-x^2)^i} - 1\right).$$

Hwang–J. (2020) solved this remaining conjecture and deduced that

$$\gamma = \frac{3}{\pi^{\frac{3}{2}}} e^{\frac{\pi^2}{24} - \frac{1}{4}} 2^{\frac{3\log 2}{2\pi^2} + 1} \approx 1.361951039.$$

Remark: It is quite difficult to guess an expression of $\gamma$.

## A two-stage saddle-point analysis

Hsien-Kuei Hwang and J. (2020) developed a two-stage saddle-point approach to directly attack the corresponding asymptotic approximations, neglecting the exactness nature of the modular form which is a strong and rare property.

- reprove Zagier's theorem (2001) on the asymptotics of the Fishburn numbers;
- reprove Bringmann-Li-Rhoades's theorem (2014) on the asymptotic number of row-Fishburn matrices;
- confirm a conjecture of Jelínek (2012) on the asymptotic number of self-dual Fishburn matrices;
- solve one open problem proposed by Jelínek (2012) and Bringmann–Li–Rhoades (2014) on the limiting distribution of Stirling statistics ($\mathcal{N}(\log n, \log n)$);
- establish the limiting distributions of several statistics with a similar sum-of-finite-product form for their g.f.s.

# Statistics on members of the Fishburn family

Fishburn matrices
dimension
1st row sum
$\#$ wNE-cells

Ascent sequences
asc, rep
zero, max
rmin

$(\mathbf{2\text{–}1})$-avoiding inversion seqs.
rep
max

$(2|\bar{3}1)$-avoiding perms.
des, iasc
lmin, lmax, rmax

$(\mathbf{2+2})$-free posets
magnitude
$\#$ minimal elements

Regular linearized chord diagrams
length of the initial run of openers
$\#$ pairs of arcs $(a, b), (c, d)$
such that $a < b = c - 1 < d - 1$

Blue: Eulerian statistics
Red: Stirling statistics

## On the limiting shape of random Fishburn matrices

Recently Hwang–J.–Schlosser (2022) find that this two-stage saddle-point approach combined with transformation formulas of basic hypergeometric series is applicable to a wider class of problems:

- (an open problem of Bringmann–Li–Rhoades 2014) Assume that all Fishburn matrices of size $n$ are equally likely to be selected. Then

$$\text{the dimension } X_n \sim \mathcal{N}\left(\frac{6n}{\pi^2}, \frac{3(12-\pi^2)n}{\pi^4}\right);$$

- (an extended open problem of Jelínek 2012) Assume that $\lambda(z)$ is a polynomial with $\lambda(1) > 1$ and that all $\lambda$-Fishburn matrices of dimension $n$ are equally likely to be selected. Then

$$\text{the size } Y_n \sim \mathcal{N}\left(\frac{\lambda'(1)}{2\lambda(1)}n^2, \frac{n^2}{2}\left(\frac{\lambda'(1)+\lambda''(1)}{\lambda(1)} - \left(\frac{\lambda'(1)}{\lambda(1)}\right)^2\right)\right);$$

- a conjecture of Stoimenow (1998).

# On the limiting shape of random Fishburn matrices

Theorem (Hwang–J. 2020, Hwang–J.–Schlosser 2022) Assume that all Fishburn matrices of size $n$ are equally likely to be selected. For a random Fishburn matrix of size $n$, we have

| | | |
|---|---|---|
| First row sum | $\mathcal{N}(\log n, \log n)$ | Stirling |
| Diagonal sum | $\mathcal{N}(2\log n, 2\log n)$ | |
| # smallest nonzero entries | $n - 2\,\mathcal{P}\text{oisson}\left(\frac{\pi^2}{6}\right)$ | |
| Dimension | $\mathcal{N}\left(\frac{6n}{\pi^2}, \frac{3(12-\pi^2)n}{\pi^4}\right)$ | Eulerian |

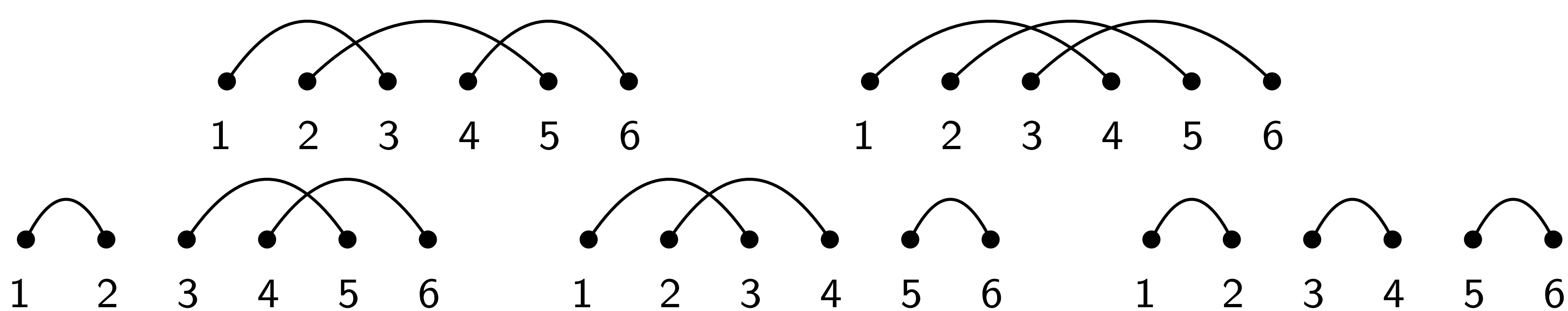
Remark: we see that in a typical random Fishburn matrix, entries equal to 1 are ubiquitous, those to 2 appear like a Poisson distribution, and the rest is asymptotically negligible.

# A conjecture of Stoimenow (1998)

Theorem (Hwang–J.–Schlosser 2022, a conjecture of Stoimenow 1998) Let $f_n$ be the number of rLCDs of size $n$ (which equals the $n$th Fishburn number), and $g_n$ be the number of connected ones of size $n$. Then,

$$g_n f_n^{-1} = e^{-1}(1 + O(n^{-1})).$$

Regular linearized chord diagrams (rLCD) is a matching of the set $[2n]$ such that it has no nested pair of arcs whose openers or the closers are next to each other.




An rLCD is connected if any arc intersects some other arcs.

## Some progress towards Stoimenow's conjecture

Theorem (Zagier 2001) The generating function

$$g(z) = \sum_{n=1}^{\infty} g_n z^n = z + z^2 + 2z^3 + 5z^4 + 16z^5 + 63z^6 + 293z^7 + \cdots$$

of connected rLCDs of size $n$ satisfies $\Phi(z, g(z)) = 1$ where

$$\Phi(z, v) = \frac{1}{1+v} \sum_{n=0}^{\infty} \prod_{i=1}^{n} \frac{1-(1-z)^i}{1+v(1-z)^i}.$$

Remark by Bringmann–Li–Rhoades (2014):

"$\Phi(z, 1)$ *is a quantum modular form related to the half-derivative of a weight* $1/2$ *modular form, similar to the situation arising for the Kontsevich's strange function*

$$F(q) = \sum_{n=0}^{\infty} \prod_{i=1}^{n} (1-q^i).$$

*What role, if any, do modular forms play in the estimation of* $g_n$*?* "

# Saddle point method

We will derive the well-known Stirling formula

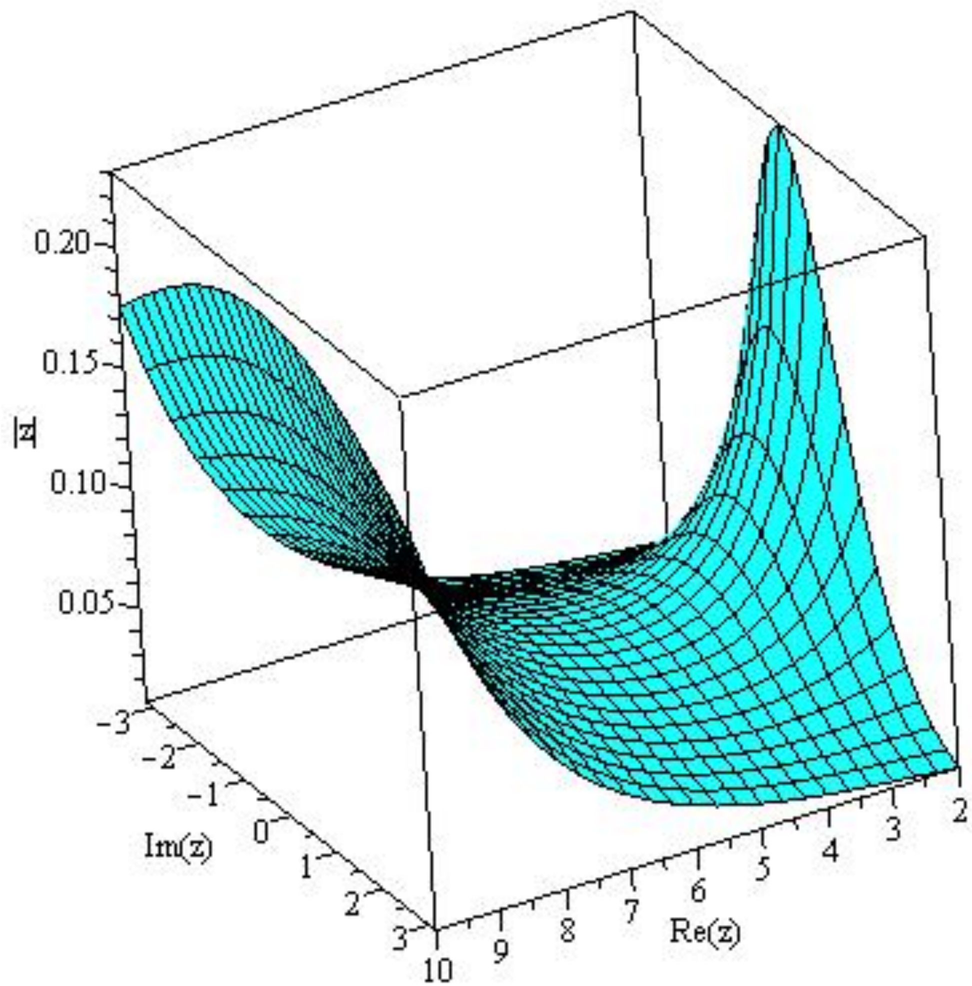
$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

via the saddle point method. This simplest case will show you how to apply the saddle point method. Our goal is to estimate the coefficients of exponential generating function $(n!)^{-1} = [z^n]exp(z)$, we start with

$$[z^n]e^z = \frac{1}{\sqrt{2\pi i}}\int_{|z|=r}\frac{e^z}{z^{n+1}}\mathrm{d}z$$

where $r$ is chosen to be the solution of $(e^z z^{-n-1})' = 0$ and $r = n+1$ is called saddle point. The saddle point corresponds locally to a maximum of the integrand along the path. It is natural to expect that a small neighbourhood of the saddle point may provide the dominant contribution to the integral.

# Saddle-point method




The modulus of the integrand $|e^z/z^{n+1}|$ for $n = 4$.

## Saddle point method

Proof sketch: Switch to polar coordinates and set $z = ne^{i\theta}$. Then,

$$[z^n]e^z = \frac{1}{\sqrt{2\pi i}} \int_{|z|=n} \frac{e^z}{z^{n+1}} \mathrm{d}z$$

$$= \frac{e^n}{n^n} \cdot \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{n(e^{i\theta}-1-i\theta)} \mathrm{d}\theta.$$

We choose $\theta_0 = n^{-2/5}$ such that $n\theta_0^2 \to \infty$ and $n\theta_0^3 \to 0$. Then

$$\int_{-\theta_0}^{+\theta_0} e^{n(e^{i\theta}-1-i\theta)} \mathrm{d}\theta = \int_{-n^{-2/5}}^{+n^{-2/5}} e^{-n\theta^2/2} \mathrm{d}\theta (1 + O(n^{-1/5}))$$

$$\sim \frac{1}{\sqrt{n}} \int_{-\infty}^{+\infty} e^{-t^2/2} \mathrm{d}t = \sqrt{\frac{2\pi}{n}} \quad \to \text{major part}$$

$$\left| \int_{\theta_0}^{2\pi-\theta_0} e^{n(e^{i\theta}-1-i\theta)} \mathrm{d}\theta \right| = O(\exp(-Cn^{1/5})) \quad \to \text{minor part}$$

# 5. Concluding remarks

# Concluding remarks

1. Two interesting generating functions on the Euler numbers:

$$\sum_{k=0}^{\infty}\prod_{j=1}^{k}\tanh(2jz)=\sum_{n=0}^{\infty}\frac{E_{2n+1}}{n!}z^n$$

$$\sum_{k=0}^{\infty}\operatorname{sech}((2k+1)z)\prod_{j=1}^{k}\tanh((2j-1)z)=\sum_{n=0}^{\infty}\frac{E_{2n}}{n!}z^n$$

These two equations are special cases of general theorems proved by Andrews–Jiménez-Urroz–Ono 2001 and Lovejoy–Ono (2003), respectively.

They have a sum-of-finite-product form. Is there a combinatorial interpretation of these two equations? An inclusion-exclusion process is expected.

# Concluding remarks

2. Recall that $\Phi(z, g(z)) = 1$ where

$$\Phi(z, v) = \frac{1}{1+v} \sum_{n=0}^{\infty} \prod_{i=1}^{n} \frac{1-(1-z)^i}{1+v(1-z)^i}.$$

Remark by Bringmann–Li–Rhoades (2014):
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$$F(q) = \sum_{n=0}^{\infty} \prod_{i=1}^{n} (1-q^i).$$

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Thank you for your attention!