Modified Macdonald polynomials and Mahonian statistics

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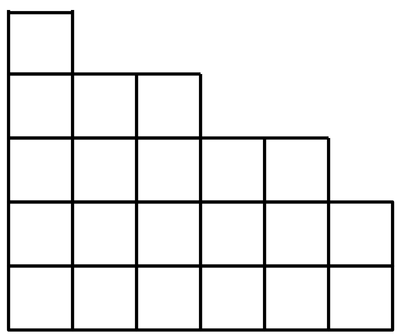
What is the modified Macdonald polynomial?

Symmetric functions

Let Λ^n denote the algebra of symmetric functions of homogeneous degree n in variables $X = \{x_1, x_2, \ldots\}$, then $\dim(\Lambda^n)$ equals the number of partitions of n.

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n is a weakly decreasing sequence of positive integers (i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$) such that the sum of λ_i's equals n.



$$\lambda = (6, 6, 5, 3, 1)$$

The bases for Λ^n: monomial symmetric functions $m_\lambda(X)$, elementary symmetric functions $e_\lambda(X)$, complete homogeneous symmetric functions $h_\lambda(X)$, power–sum symmetric functions $p_\lambda(X)$ and Schur functions $s_\lambda(X)$.

Macdonald polynomials

Macdonald polynomials $P_\lambda(X; q, t)$ indexed by partitions are polynomials in infinitely many variables $X = \{x_1, x_2, \ldots\}$ with coefficients in the field $Q(q, t)$ of rational functions of two variables q and t. They are defined as the unique basis for the ring of symmetric functions over the field $Q(q, t)$ with orthogonal property and lower triangular property, namely,

$$P_\lambda(X; q, t) = m_\lambda(X) + \sum_{\mu < \lambda} c_{\lambda\mu}(q, t) m_\mu(X)$$

for $c_{\lambda\mu}(q, t) \in Q(q, t)$. Here $m_\lambda(X)$ is the monomial symmetric function. Furthermore, $\langle P_\lambda, P_\mu \rangle = \delta_{\lambda\mu}$ with respect to the inner product

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu} \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}$$

where $z_\lambda = 1^{k_1} k_1! 2^{k_2} k_2! \cdots$ and i appears k_i times in λ.

Macdonald polynomials

Example: For $n = 2$, there are two partitions $(1, 1)$ and (2) of 2, satisfying $(1, 1) < (2)$ by the dominance order. Since

$$P_\lambda(X; q, t) = m_\lambda(X) + \sum_{\mu<\lambda} c_{\lambda\mu}(q, t) m_\mu(X),$$

we have $P_{11}(X; q, t) = m_{11}(X)$. Let $c_{2,11}(q, t) = c$, then

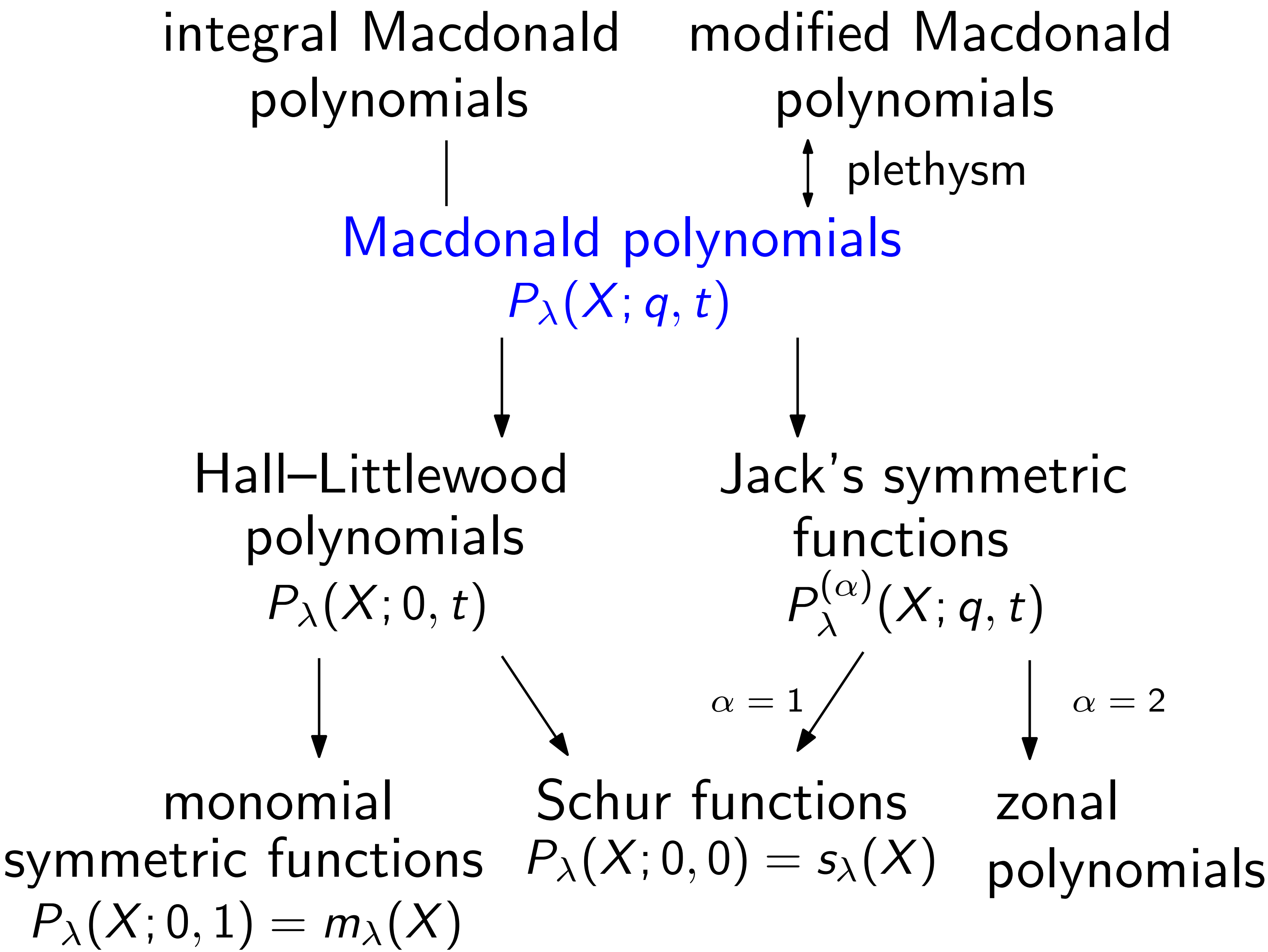
$$\begin{aligned} P_2(X; q, t) &= m_2(X) + c m_{11}(X), \\ &= \frac{c}{2} p_{11}(X) + (1 - \frac{c}{2}) p_2(X), \end{aligned}$$

where c is determined by the condition that $\langle P_2, P_{11} \rangle = 0$. That is,

$$\langle P_{11}, P_2 \rangle = \left\langle \frac{1}{2} p_{11} - \frac{1}{2} p_2, \frac{c}{2} p_{11} + (1 - \frac{c}{2}) p_2 \right\rangle = 0,$$

which gives that

$$P_2(X; q, t) = m_2(X) + \frac{(1 + q)(1 - t)}{1 - qt} m_{11}(X).$$

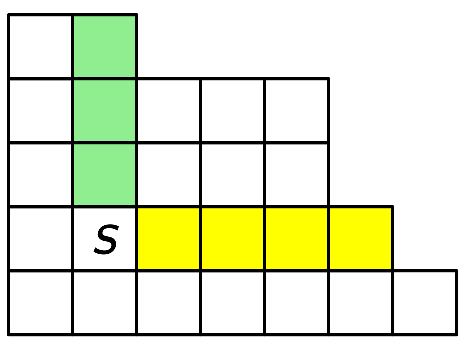

integral Macdonald polynomials
modified Macdonald polynomials
plethysm
Macdonald polynomials
$P_\lambda(X;q,t)$
Hall–Littlewood polynomials
$P_\lambda(X;0,t)$
Jack's symmetric functions
$P_\lambda^{(\alpha)}(X;q,t)$
$\alpha = 1$
$\alpha = 2$
monomial symmetric functions
$P_\lambda(X;0,1) = m_\lambda(X)$
Schur functions
$P_\lambda(X;0,0) = s_\lambda(X)$
zonal polynomials

Integral Macdonald polynomials

Since the coefficients in this expansion have nontrivial denominators, Macdonald proposed the integral form of $P_\lambda(X;q,t)$, denoted by $J_\lambda(X;q,t)$, named as the integral Macdonald polynomials.

$$J_\lambda(X;q,t) = \prod_{s\in D(\lambda)} (1 - q^{arm(s)} t^{1+leg(s)}) P_\lambda(X;q,t)$$

where $arm(s)$ and $leg(s)$ are the number of boxes strictly east of s, and north of s respectively.



$$\lambda = (7,6,5,5,2),$$
$$leg(s) = 3,$$
$$arm(s) = 4.$$

Macdonald conjectured that the coefficients of $J_\lambda(X;q,t)$ are polynomials in q and t. This is called integrality conjecture.

Integral Macdonald polynomials

The integrality conjecture was affirmed by Garsia and Remmel (1998), Garsia and Testler (1996), Kirillov and Noumi (1998), Knop (1997), Sahi (1996). For instance,

$$P_{22}(X;q,t) = m_{22}(X) + \frac{(1+q)(1-t)}{1-qt} m_{211}(X) + \frac{(2+t+3q+q^2+3qt+2q^2t)(1-t)^2}{(1-qt)(1-qt^2)} m_{1111}(X).$$

$$\begin{aligned} J_{22}(X;q,t) &= (1-qt)(1-t)(1-qt^2)(1-t^2)P_{22}(X;q,t), \\ &= (1-qt)(1-t)(1-qt^2)(1-t^2)m_{22}(X) \\ &\quad +(1-t)^2(1-qt^2)(1-t^2)(1+q)m_{211}(X) \\ &\quad +(2+t+3q+q^2+3qt+2q^2t) \\ &\quad \times(1-t)^3(1-t^2)m_{1111}(X). \end{aligned}$$

Modified Macdonald polynomials

Macdonald positive conjecture says that the coefficient $K_{\lambda\mu}(q,t)$ defined by the Schur expansion of integral Macdonald polynomials

$$J_\mu(X;q,t) = \sum_\mu K_{\lambda\mu}(q,t) s_\lambda[X(1-t)]$$

is a polynomial with non-negative coefficients where $f[X]$ denotes the plethystic substitution of X into the symmetric function f.

Macdonald positive conjecture was confirmed by Haiman (2001) through working with modified Macdonald polynomials (introduced by Garsia and Haiman in 1993), defined as

$$\tilde{H}_\mu(X;q,t) = t^{n(\mu)} J_\mu\left[\frac{X}{1-t^{-1}}; q, t^{-1}\right] = \sum_\mu \tilde{K}_{\lambda\mu}(q,t) s_\lambda(X).$$

Here $\tilde{K}_{\lambda\mu}(q,t) = t^{n(\mu)} K_{\lambda\mu}(q,t^{-1})$. Since $K_{\lambda\mu}(q,t)$ has degree at most $n(\mu)$ in t, the coefficient $\tilde{K}_{\lambda\mu}(q,t) \in N[q,t]$ if and only if $K_{\lambda\mu}(q,t) \in N[q,t]$.

Modified Macdonald polynomials

Example: For $\mu = (2,2)$,

$$\begin{aligned}
J_{22}(X;q,t) &= (1-qt)(1-t)(1-qt^2)(1-t^2)m_{22}(X) \\
&\quad +(1-t)^2(1-qt^2)(1-t^2)(1+q)m_{211}(X) \\
&\quad +(2+t+3q+q^2+3qt+2q^2t) \\
&\qquad \times(1-t)^3(1-t^2)m_{1111}(X). \\
&= t^2 s_4[X(1-t)] + (qt^2+t+qt)s_{31}[X(1-t)] \\
&\quad +(1+q^2t^2)s_{22}[X(1-t)] + q^2 s_{1111}[X(1-t)] \\
&\quad +(q+q^2t+qt)s_{211}[X(1-t)].
\end{aligned}$$

$$\begin{aligned}
\tilde{H}_{22}(X;q,t) &= s_4(X) + (q+t+qt)s_{31}(X) + (q^2+t^2)s_{22}(X) \\
&\quad +(qt^2+q^2t+qt)s_{211}(X) + q^2t^2 s_{1111}(X).
\end{aligned}$$

Combinatorial formulas
of modified Macdonald polynomials

Modified Macdonald polynomials

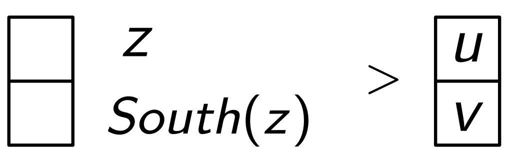
Theorem (Haglund, Haiman, Loehr, 2005):

$$\tilde{H}_\lambda(X;q,t) = \sum_{\sigma\colon D(\lambda)\to P} x^\sigma t^{maj(\sigma)} q^{inv(\sigma)},$$

where $x^\sigma = \prod_{z\in D(\lambda)} x_{\sigma(z)} = x_1^{\#1's} x_2^{\#2's} \cdots x_i^{\#i's} \cdots$ and the statistics *maj* (the major index), *inv* (inversion) are natural extensions of classical permutation statistics. Define

$$Des(\sigma) = \{z \in D(\lambda) : \sigma(z) > \sigma(South(z))\},$$
$$maj(\sigma) = \sum_{z\in Des(\sigma)} (\text{leg}(z)+1),$$

to be the descent set and the major index of σ, respectively.



$\sigma(z) = u$

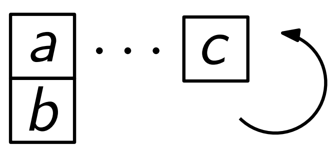
$\sigma(South(z)) = v$

Modified Macdonald polynomials

Theorem (Haglund, Haiman, Loehr, 2005):

$$\tilde{H}_\lambda(X;q,t) = \sum_{\sigma\colon D(\lambda)\to P} x^\sigma t^{maj(\sigma)} q^{inv(\sigma)}.$$

For $\sigma \in F(\lambda)$, add a box with entry ∞ below the bottommost box of each column of σ. Let $inv(\sigma)$ count the number of inversion triples where an inversion triple is a triple (a, b, c) of entries such that



and

$a < b < c$,
$b < c < a$,
$c < a < b$ or $a = b \neq c$.

$\sigma =$

5	4					
9	3	6	1	3		
2	5	9	4	8		
3	9	7	3	5		
5	8	4	6	4	8	7
9	3	6	5	2	10	1
∞	∞	∞	∞	∞	∞	∞

5	4
9	

is an inversion triple.

9		1
2		

is not an inversion triple.

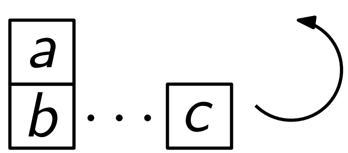
Modified Macdonald polynomials

Theorem (Corteel, Haglund, Mandelshtam, Mason and Williams, 2021):

$$\tilde{H}_\lambda(X; q, t) = \sum_\sigma perm(\sigma) x^\sigma t^{maj(\sigma)} q^{inv(\sigma)}$$

summed over sorted tableaux, where $perm(\sigma)$ is a q-multinomial. Interestingly, a new statistic *quinv* was introduced, inspired by multiline queue formula to compute the stationary probabilities of the ASEP due to Martin (2020).

Definition: Given a filling σ, add a box with entry 0 above the topmost box of each column of σ. Let $quinv(\sigma)$ count queue inversion triples where a queue inversion triple is a triple (a, b, c) of entries such that



and

$a < b < c$,
$b < c < a$,
$c < a < b$ or $a = b \neq c$.

Modified Macdonald polynomials

Theorem (Ayyer, Mandelshtam, Martin, 2023):

$$\tilde{H}_\lambda(X; q, t) = \sum_{\sigma\colon D(\lambda)\to P} x^\sigma t^{maj(\sigma)} q^{quinv(\sigma)}.$$

This was conjectured by Corteel, Haglund, Mandelshtam, Mason and Williams (2021). Proof strategy: Verify that the RHS satisfies the conditions that uniquely characterize the modified Macdonald polynomials:

$$\begin{aligned} &(1) \quad \tilde{H}_\mu[X(1-q); q, t] \in Q(q,t)\{s_\lambda : \lambda \geq \mu\}; \\ &(2) \quad \tilde{H}_\mu[X(1-t); q, t] \in Q(q,t)\{s_\lambda : \lambda \geq \mu'\}; \\ &(3) \quad \langle \tilde{H}_\mu, s_n \rangle = 1. \end{aligned}$$

Beyond that, Ayyer, Mandelshtam and Martin (2023) proposed a conjecture on the refined equivalence of these two formulas.

Modified Macdonald polynomials

Two fillings σ, τ of $D(\lambda)$ are row-equivalent if the multisets of entries in the ith row of σ and τ are exactly the same for all i.

Theorem (J. and Lin, 2024): Let $[\sigma]$ denote the row-equivalent class of σ, then

$$\sum_{\tau\in[\sigma]} t^{maj(\tau)} q^{inv(\tau)} = \sum_{\tau\in[\sigma]} t^{maj(\tau)} q^{quinv(\tau)}.$$

If σ is a filling of a rectangular diagram, then

$$\sum_{\tau\in[\sigma]} t^{maj(\tau)} q^{inv(\tau)} u^{quinv(\tau)} = \sum_{\tau\in[\sigma]} t^{maj(\tau)} u^{inv(\tau)} q^{quinv(\tau)}.$$

Remark: The second equation is not generally true, though it also holds for column strict fillings σ, proved by Bhattacharya, Ratheesh and Viswanath (2023, 2024). Their proofs are bijective, which develop novel connections between different combinatorial models, maps and statistics.

Modified Macdonald polynomials

Example: Consider the filling σ and its row-equivalent class $[\sigma]$:

$[\sigma]$	$\begin{smallmatrix}3&&\\4&1&2\\3&3&3\end{smallmatrix}$	$\begin{smallmatrix}3&&\\4&2&1\\3&3&3\end{smallmatrix}$	$\begin{smallmatrix}3&&\\1&2&4\\3&3&3\end{smallmatrix}$	$\begin{smallmatrix}3&&\\1&4&2\\3&3&3\end{smallmatrix}$	$\begin{smallmatrix}3&&\\2&4&1\\3&3&3\end{smallmatrix}$	$\begin{smallmatrix}3&&\\2&1&4\\3&3&3\end{smallmatrix}$
maj	2	2	2	2	2	2
inv	0	1	2	1	2	3
quinv	3	2	2	1	0	1

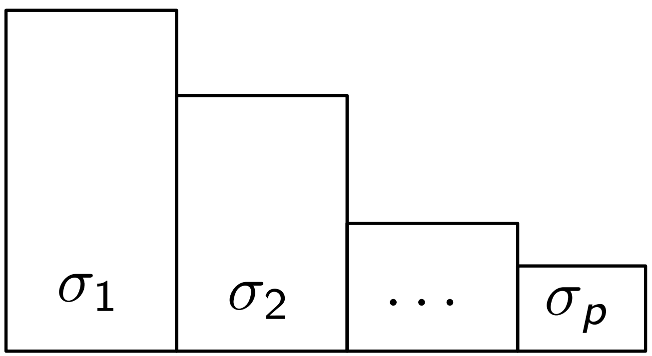
$$\sum_{\tau\in[\sigma]} t^{maj(\tau)} q^{inv(\tau)} u^{quinv(\tau)} \neq \sum_{\tau\in[\sigma]} t^{maj(\tau)} u^{inv(\tau)} q^{quinv(\tau)}$$

for the non-rectangular diagram of $\lambda = (3,3,1)$.

The proofs: two operators on fillings

Reverse operator

Each Young diagram $D(\lambda)$ is regarded as a concatenation of maximal rectangles in a way that the heights of rectangles are strictly decreasing from left to right. Let $\sigma = \sigma_1 \sqcup \cdots \sqcup \sigma_p$ where p is the number of rectangles of $D(\lambda)$.



9	3	6	1	3
2	5	9	4	8
3	9	7	3	5
5	8	4	6	4
9	3	6	5	2

→

3	1	6	3	9
8	4	9	5	2
5	3	7	9	3
4	6	4	8	5
2	5	6	3	9

For a partition λ and a filling $\sigma = \sigma_1 \sqcup \cdots \sqcup \sigma_p$, define $rev(\sigma) = rev(\sigma_1) \sqcup \cdots \sqcup rev(\sigma_p)$ as the reverse of σ where the filling $rev(sigma)$ is obtained by reversing the sequence of entries of each row.

Flip operator

For a filling $\sigma \in F(\lambda)$, an index i such that $\lambda'_i = \lambda'_{i+1}$ and an integer $r \leq \lambda'_i$, let $t_i^{(r)}$ be the operator that acts on σ by interchanging the entries $\sigma(r,i)$ and $\sigma(r,i+1)$. For $1 \leq s \leq r \leq \lambda'_i$, let

$$t_i^{[s,r]} := t_i^{(s)} \circ t_i^{(s+1)} \circ \cdots \circ t_i^{(r)}$$

denote the flip operator that swaps entries $\sigma(x,i)$ and $\sigma(x,i+1)$ for all x with $s \leq x \leq r$. The flip operator ρ_i^r is defined as follows:

1. if $\sigma(x,i) = \sigma(x,i+1)$ for all $1 \leq x \leq r$, then $\rho_i^r(\sigma) = \sigma$;
2. otherwise, let k be the largest integer such that $k \leq r$ and $\sigma(k,i) \neq \sigma(k,i+1)$. Let h be the largest integer such that $h \leq k$, $\sigma(h,i) \neq \sigma(h,i+1)$ and

$$\begin{matrix} \sigma(h,i) & \\ \sigma(h-1,i) & \sigma(h-1,i+1) \end{matrix} \qquad \begin{matrix} \sigma(h,i+1) & \\ \sigma(h-1,i) & \end{matrix} \sigma(h-1,i+1)$$

are both queue inversion triples or both are not. Define $\rho_i^r = t_i^{[h,k]}$.

Flip operator

Example: For the filling σ as below, $\rho_1^6(\sigma) = t_1^{[3,5]}(\sigma)$ is generated as follows. Since row 5 is the topmost row with different entries, $k = 5$ and ρ_1^6 starts from this row. Further, ρ_1^6 terminates at row 3 ($h = 3$), as triples $(3, 5, 8)$ and $(9, 5, 8)$ are both queue inversion triples.

	σ		→		→		→	
row 6	3	3	3	3	3	3	3	3
row 5	9	3	3	9	3	9	3	9
row 4	2	5	2	5	5	2	5	2
row 3	3	9	3	9	3	9	9	3
row 2	5	8	5	8	5	8	5	8
row 1	9	3	9	3	9	3	9	3

Remark: Loehr and Niese (2012) introduced the column switch operator to describe the change of inversions (inv). In parallel, Ayyer, Mandelshtam and Martin (2023) defined the flip operator for quinv. We unify these two operators to construct the desired bijection.

The bijective proof outline

Goal: A bijection $\varphi : F(\lambda) \to F(\lambda)$ satisfying $\varphi(\sigma) \sim \sigma$ and

$$(quinv, maj)(\varphi(\sigma)) = (inv, maj)(\sigma).$$

In particular, if $D(\lambda)$ is a rectangle, then

$$(inv, quinv, maj)(\varphi(\sigma)) = (quinv, inv, maj)(\sigma).$$

Let $\sigma = \sigma_1 \sqcup \cdots \sqcup \sigma_p$, define

$$\kappa(\sigma) := \sum_{i=1}^{p} (quinv(\sigma_i) - inv(\sigma_i^r)).$$

Theorem: There is a bijection $\gamma : F(\lambda) \to F(\lambda)$ satisfying $\gamma(\sigma) \sim \sigma$,

$$quinv(\gamma(\sigma)) = inv(\sigma) + \kappa(\gamma(\sigma)),$$
$$maj(\gamma(\sigma)) = maj(\sigma),$$
$$\mathcal{N}des(\sigma_1) = \mathcal{N}des((\gamma(\sigma)_1)^r),$$

and the topmost rows of σ and $\gamma(\sigma)$ are reverse of each other. Here $\mathcal{N}des(\sigma) = (x_1, \ldots, x_k)$ and x_i counts non-descents of column i.

The bijective proof outline

The second bijection $\theta : F(\lambda) \to F(\lambda)$ acts on each rectangle of the fillings independently and decreases the number of queue inversions by $\kappa(\gamma(\sigma))$ but preserves the major index, by which we find the desired bijection φ.

Both bijections γ, θ are constructed by the involution ϕ_i:

Theorem: For a partition λ and $\lambda'_i = \lambda'_{i+1}$, let $\sigma \in F(\lambda)$ and x_i be the number of non-descents in the ith column of σ. Then there is an involution $\phi_i : F(\lambda) \to F(\lambda)$ such that $\phi_i(\sigma) \sim \sigma$, and for $\nu \in \{inv, quinv\}$,

$$maj(\phi_i(\sigma)) = maj(\sigma),$$
$$\nu(\phi_i(\sigma)) = \nu(\sigma) + x_{i+1} - x_i,$$
$$\mathcal{N}des(\phi_i(\sigma)) = s_i \circ \mathcal{N}des(\sigma),$$

where $s_i \circ (\ldots x_i, x_{i+1} \ldots) = (\ldots x_{i+1}, x_i \ldots)$.

An example of this bijection

$\sigma =$

5	4					
9	3	6	1	3		
2	5	9	4	8		
3	9	7	3	5		
5	8	4	6	4	8	7
9	3	6	5	2	10	1

0	0	5	3	7	9	3
7	8	4	6	4	8	5
1	10	2	5	6	3	9

→

0	5	0	3	7	9	3
7	4	8	6	4	8	5
1	2	10	5	6	3	9

→

0	5	3	0	7	9	3
7	4	8	6	4	8	5
1	2	5	10	6	3	9

→

0	5	3	7	0	9	3
7	4	8	6	4	8	5
1	2	5	10	6	3	9

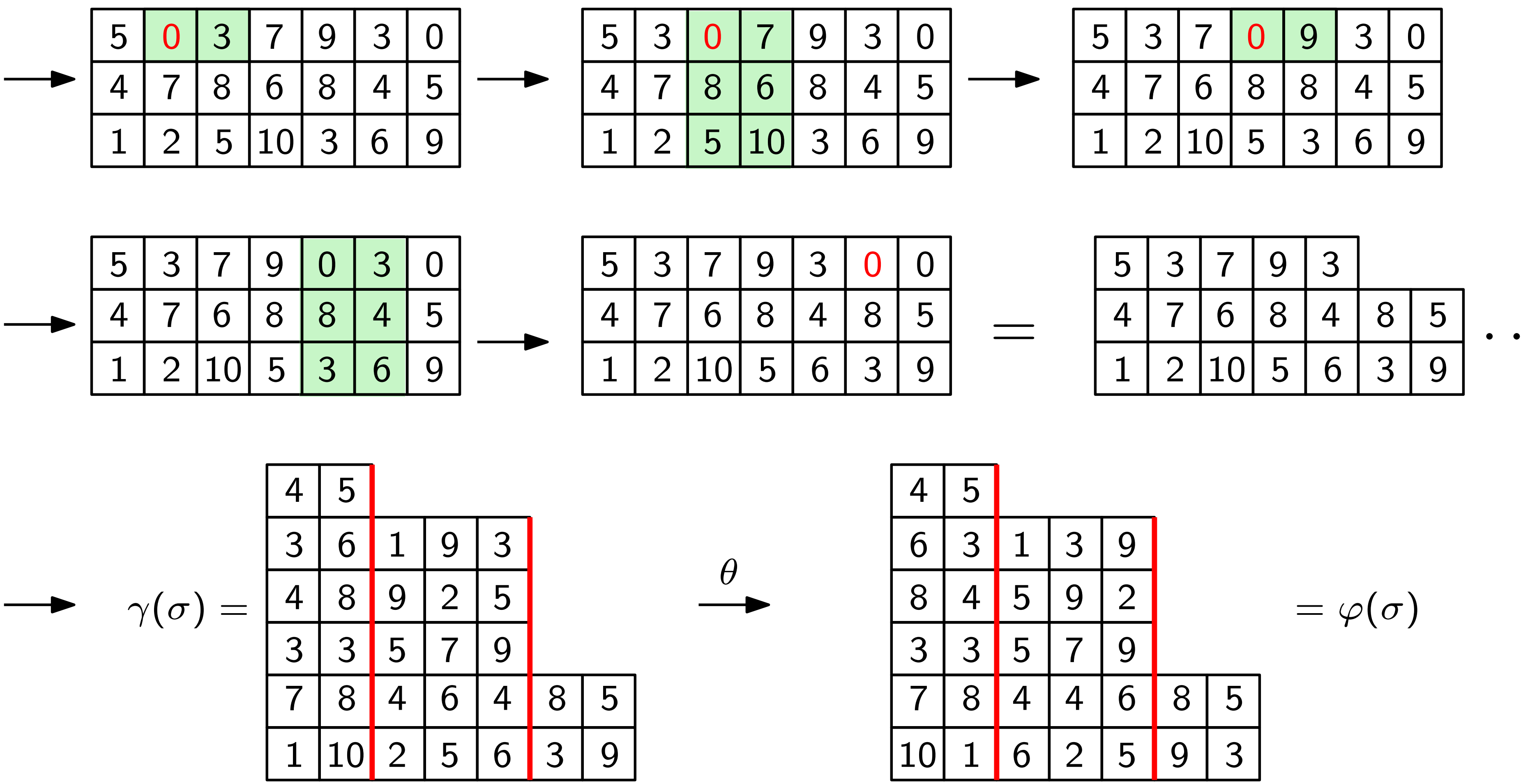
→

0	5	3	7	9	0	3
7	4	8	6	8	4	5
1	2	5	10	3	6	9

→

0	5	3	7	9	3	0
7	4	8	6	8	4	5
1	2	5	10	3	6	9

An example of this bijection



Some related works

Some related works

Theorem (Mandelshtam, 2023)

$$\tilde{H}_\lambda(X;q,t) = \sum_\sigma perm(\sigma) x^\sigma t^{maj(\sigma)} q^{quinv(\sigma)}$$

summed over *quinv*–sorted tableaux. On the other hand, through an integrable lattice model construction and matrix product formula for Macdonald polynomials by Cantini, de Gier and Wheeler (2015),

Theorem (Garbali and Wheeler, 2020)

$$\tilde{H}_\lambda(X;q,t) = q^{n(\lambda')} \sum_\mu \mathcal{P}_{\lambda\mu}(t,q^{-1}) m_\mu(X), \text{ where}$$

$$\mathcal{P}_{\lambda\mu}(t,q) = \sum_{\{\nu\}} q^{\chi_1(\nu)} \prod_{i \le j} \Phi_{\nu_{i+1,j}|\nu_{i,j}}(t^{j-i} q^{\lambda_i - \lambda_j}; q)$$

is a manifestly positive polynomial in q and t.

Example: For $\lambda = (2,2,1)$ and $\mu = (4,1)$,

$$\mathcal{P}_{\lambda\mu}(t,q) = q^2 + q^3 + tq^3 + q^4 + tq^4.$$

Some related works

Theorem (J. and Lin, 2025): Let $[\sigma]$ denote the row-equivalent class of σ, and let A be the family of new statistics. Then for every $\eta \in A$,

$$\sum_{\tau \in [\sigma]} t^{maj(\tau)} q^{\eta(\tau)} = \sum_{\tau \in [\sigma]} t^{maj(\tau)} q^{quinv(\tau)}.$$

Here $A \cap \{inv, quinv\} = \emptyset$ and A contains 16 statistics.

Theorem (J. and Lin, 2025): Let $\mathcal{C}_\circ(\lambda)$ and $\mathcal{C}_\dagger(\lambda)$ be the sets of canonical and dual canonical tableaux of the Young diagram of λ, respectively. Then for $\varepsilon \in \{\circ, \dagger\}$ and four statistics $\eta \in A$,

$$\tilde{H}_\lambda(X; q, t) = \sum_{\sigma \in \mathcal{C}_\varepsilon(\lambda)} d_\varepsilon(\sigma) x^\sigma t^{maj(\sigma)} q^{\eta(\sigma)},$$

where $d_\varepsilon(\sigma)$ is a t-multinomial. These four compact formulas will produce four explicit expressions of $\mathcal{P}_{\lambda\mu}(t, q)$. One of them is consistent with the one by Garbali and Wheeler (2020).

Thank you!

https://arxiv.org/pdf/2407.14099

https://arxiv.org/pdf/2506.23373