MODIFIED MACDONALD POLYNOMIALS AND MAHONIAN STATISTICS

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ABSTRACT. The celebrated Haglund–Haiman–Loehr theorem (2005) provides a combinatorial formula for the modified Macdonald polynomials.

$$\tilde{H}_\lambda(X;q,t) = \sum_{\sigma\in\mathcal{T}(\lambda)} x^\sigma q^{\mathsf{maj}(\sigma)} t^{\mathsf{inv}(\sigma)}.$$

Inspired by Martin's multiline queue formula (2018) for the stationary distribution of multitype asymmetric simple exclusion processes, Corteel, Haglund, Mandelshtam, Mason and Williams (2019) considered a queue inversion statistic quinv and conjectured that $\tilde{H}_\lambda(X;q,t)$ is invariant if the inversion statistic inv is replaced by quinv. This was subsequently resolved by Ayyer, Mandelshtam and Martin (2021) and a stronger conjecture on the equivalence of two refined formulas for $\tilde{H}_\lambda(X;q,t)$ was proposed by them.

Our main result affirms the Ayyer–Mandelshtam–Martin conjecture. That is, we establish an equidistribution between the pairs $(\mathsf{inv},\mathsf{maj})$ and $(\mathsf{quinv},\mathsf{maj})$ on any row-equivalency class $[\tau]$ where τ is a filling of a given Young diagram. As a byproduct of our approach, we find that if τ is a filling of a rectangular diagram, the triples $(\mathsf{inv},\mathsf{quinv},\mathsf{maj})$ and $(\mathsf{quinv},\mathsf{inv},\mathsf{maj})$ have the same distribution over $[\tau]$.

1. Introduction

Macdonald polynomials $P_\mu(X;q,t)$ indexed by partitions are polynomials in infinitely many variables $X=\{x_1,x_2,\ldots\}$ with coefficients in the field $\mathbb{Q}(q,t)$ of rational functions of two variables q and t. Several important classes of symmetric polynomials are well–studied specializations of Macdonald polynomials such as Schur polynomials (when $q=t$), Hall–Littlewood polynomials (when $q=0$) and Jack polynomials (when $q=t^\alpha$ and let $t\to 1$).

Macdonald polynomials $P_\mu(X;q,t)$ are defined as the unique basis for the ring of symmetric functions over the field $\mathbb{Q}(q,t)$ with orthogonal property and lower triangular property. The former is defined through the Hall scalar product and the latter by an expansion of $P_\lambda(X;q,t)$ with respect to monomial symmetric functions $m_\lambda(X)$. Since the coefficients in this expansion have nontrivial denominators, Macdonald introduced the *integral form* of $P_\lambda(X;q,t)$, denoted by $J_\lambda(X;q,t)$, which is defined as

$$J_\mu(X;q,t) = \sum_\lambda K_{\lambda\mu}(q,t) s_\lambda[X(1-t)], \tag{1.1}$$

where $f[X]$ denotes the plethystic substitution of X into the symmetric function f, s_λ is the Schur function and $K_{\lambda\mu}(q,t)$ is the q,t-Kostka numbers. Subsequently, another widely

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studied variant of Macdonald polynomials, called *modified Macdonald polynomials* $\tilde{H}_\mu(X;q,t)$ was introduced by Garsia and Haiman [6]. Let $\tilde{K}_{\lambda\mu}(q,t) = t^{n(\lambda)}K_{\lambda\mu}(q,t^{-1})$ where $n(\lambda) = \sum_i (i-1)\lambda_i$. Then

$$\tilde{H}_\mu(X;q,t) = \sum_\lambda \tilde{K}_{\lambda\mu}(q,t)s_\lambda(X).$$

Haiman remarkably found that $\tilde{H}_\lambda(X;q,t)$ equals the Frobenius series of a space as the linear span of certain polynomials and their all partial derivatives [9]. In parallel, the combinatorial investigation of modified Macdonald polynomials has been greatly promoted by the celebrated breakthrough on the surprising connections between $\tilde{H}_\lambda(X;q,t)$ and permutation statistics on fillings of Young diagrams due to Haglund, Haiman and Loehr [10]. In the sequel, we represent a Young diagram in a French manner.

Let $\mathcal{T}(\lambda)$ denote the set of fillings of the Young diagram of λ such that each cell is filled with a positive integer, and let x^σ be the product of x_i's whenever i is an entry of σ. Then Haglund, Haiman and Loehr [10] established that

$$\tilde{H}_\lambda(X;q,t) = \sum_{\sigma\in\mathcal{T}(\lambda)} x^\sigma q^{\mathsf{maj}(\sigma)}t^{\mathsf{inv}(\sigma)} \tag{1.2}$$

where $\mathsf{maj}(T)$ and $\mathsf{inv}(T)$ are natural extensions, respectively of the major index and inversion number for permutations. The precise definitions are postponed to Section 2. Very Recently, Corteel, Haglund, Mandelshtam, Mason and Williams [4, 5] discovered a compact formula for $\tilde{H}_\lambda(X;q,t)$ which is a sum over sorted tableaux and made a conjecture on an equivalent form of (1.2):

$$\tilde{H}_\lambda(X;q,t) = \sum_{\sigma\in\mathcal{T}(\lambda)} x^\sigma q^{\mathsf{maj}(\sigma)}t^{\mathsf{quinv}(\sigma)} \tag{1.3}$$

where quinv is called *queue inversion* and to be defined in Section 2. This conjecture was confirmed by Ayyer, Mandelshtam and Martin [1] by proving that the RHS of (1.3) satisfies certain orthogonal and triangular conditions which uniquely determine the modified Macdonald polynomials $\tilde{H}_\lambda(X;q,t)$.

Interestingly, a refinement of the equivalence between (1.2) and (1.3) was conjectured by Ayyer, Mandelshtam and Martin (Conjecture 10.3 of [1]). Our main result is an affirmation of this conjecture; see (1.4) Theorem 1. As a bonus of our approach, we find the equidistribution (1.5) between the triples $(\mathsf{inv},\mathsf{quinv},\mathsf{maj})$ and $(\mathsf{quinv},\mathsf{inv},\mathsf{maj})$ for all rectangular diagrams. To be precise, we call two fillings σ,τ of $\mathsf{dg}(\lambda)$ *row-equivalent* if the multisets of entries in the ith row of σ and τ are exactly the same for all i. The precise statement of our main result is the following.

Theorem 1. *Let $[\sigma]$ denote the row-equivalent class of σ, then*

$$\sum_{\tau\in[\sigma]} q^{\mathsf{maj}(\tau)}t^{\mathsf{inv}(\tau)} = \sum_{\tau\in[\sigma]} q^{\mathsf{maj}(\tau)}t^{\mathsf{quinv}(\tau)}. \tag{1.4}$$

If σ is a filling of a rectangular diagram, then

$$\sum_{\tau\in[\sigma]} q^{\mathsf{maj}(\tau)}t^{\mathsf{inv}(\tau)}u^{\mathsf{quinv}(\tau)} = \sum_{\tau\in[\sigma]} q^{\mathsf{maj}(\tau)}u^{\mathsf{inv}(\tau)}t^{\mathsf{quinv}(\tau)}. \tag{1.5}$$

It is worth pointing out that the symmetric distribution (1.5) is not true for arbitrary filling σ and we provide such an example in Remark 2.

Three subsets of $[\sigma]$, respectively with extreme values of the major index or (queue) inversion numbers are shown to satisfy (1.4) by Bhattacharya, Ratheesh and Viswanath [2, 3]. Their proofs are bijective, which develop novel connections between different combinatorial models, maps and statistics such that Gelfand–Tsetlin patterns, partitions overlaid patterns, box complementation [2] and charge and cocharge on words [3].

We take a different approach, highlighting that the reverse operator and a column switch operator are sufficient to prove Theorem 1. The rest of the paper is organized as follows: In Section 2 we present preliminaries on the Mahonian statistics inv, quinv and maj of fillings. In Section 3 the reverse operator and flip operator as the starting point of our proof are described. Section 4 shows a roadmap of our proof and Sections 5–6 are devoted to proving Theorem 1.

2. Preliminaries and notations

A partition $\lambda = (\lambda_1, \cdots, \lambda_k)$ of n is sequence of positive integers such that $\lambda_i \geq \lambda_{i+1}$ for all $1 \leq i < k$ and $|\lambda| = \lambda_1 + \cdots + \lambda_k = n$. Each λ_i is called the ith part of λ and k is the length of λ, denoted by $\ell(\lambda)$. The Young diagram of λ, denoted by $\mathsf{dg}(\lambda)$, is an array of boxes with λ_i boxes in the ith row from bottom to top, with the first box in each row left-justified. A box has coordinate (i, j) if it is in the ith row from bottom to top and jth column from left to right. Let λ' be the transpose of λ, that is, $\mathsf{dg}(\lambda')$ is obtained from $\mathsf{dg}(\lambda)$ by reflecting across the main diagonal (boxes with coordinate (i, i)).

A filling of $\mathsf{dg}(\lambda)$ is a function $\sigma : \mathsf{dg}(\lambda) \to \mathbb{P}$ ($\mathbb{P}$ is the set of positive integers), which assigns each box u of $\mathsf{dg}(\lambda)$ to a positive integer $\sigma(u)$. We use $\mathsf{South}(u)$ to denote the box right below u. Let $\mathcal{T}(\lambda)$ denote the set of all fillings of $\mathsf{dg}(\lambda)$, and set

$$x^\sigma = \prod_{u \in \mathsf{dg}(\lambda)} x_{\sigma(u)}$$

be the monomial of σ. A *descent* (or *non-descent*) of a filling $\sigma \in \mathcal{T}_\lambda$ is a pair of entries $(\sigma(x), \sigma(\mathsf{South}(x))$ such that $\sigma(x) > \sigma(\mathsf{South}(x))$ (or $\sigma(x) \leq \sigma(\mathsf{South}(x))$). Define $\mathsf{Des}(\sigma) = \{x \in \mathsf{dg}(\lambda) : (\sigma(x), \sigma(\mathsf{South}(x)) \text{ is a descent}\}$ to be the *descent set* of σ and $\mathsf{des}(\sigma) = |\mathsf{Des}(\sigma)|$. Let $\mathsf{leg}(u)$ be the number of boxes strictly above u in its column, then

$$\mathsf{maj}(\sigma) = \sum_{u \in \mathsf{Des}(\sigma)} (\mathsf{leg}(u) + 1)$$

is called *the major index* of σ. Define $\mathcal{N}\mathsf{des}(\sigma) = (a_1, \ldots, a_k)$ where a_i counts the number of non-descents in column i of σ and set $\mathsf{ndes}(\sigma) = a_1 + \cdots + a_k$ be the number of non-descents of σ.

Given a filling σ, let $\hat{\sigma}$ be the filling obtained by adding a box with entry 0 above the topmost box of each column of σ. A *queue inversion triple* of σ is a triple (a, b, c) of entries in $\hat{\sigma}$ such that (as shown below on the left)

(1) b and c are in the same row and c is to the right of b;

(2) a and b are in the same column such that b is right below a;
(3) one of the conditions $a<b<c$, $b<c<a$, $c<a<b$ and $a=b\neq c$ is true.

Let $\tilde{\sigma}$ be the filling obtained by adding a box with entry ∞ below the bottommost box of each column of σ. An *inversion triple* of σ is a triple (a,b,c) of entries in $\tilde{\sigma}$ satisfying the above (2)–(3) and (4), as shown above on the right.

(4) a and c are in the same row and c is to the right of a.

Define $\mathcal{Q}(a,b,c)=1$ if (a,b,c) is a queue inversion triple or an inversion triple; otherwise $\mathcal{Q}(a,b,c)=0$. Let $\mathsf{quinv}(\sigma)$ and $\mathsf{inv}(\sigma)$ be the numbers of queue inversion triples and inversion triples of σ, respectively. Following the same strategy from [14], one is able to prove Theorem 1 for $q=1$, i.e.,

$$\sum_{\tau\in[\sigma]} t^{\mathsf{inv}(\tau)} = \sum_{\tau\in[\sigma]} t^{\mathsf{quinv}(\tau)} = \prod_{i=1}^{\ell(\lambda)} \begin{bmatrix} a_{i1}+\cdots+a_{iN} \\ a_{i1},\ldots,a_{iN} \end{bmatrix}_t, \tag{2.1}$$

where the ith row of σ consists of a_{i1} copies of 1, a_{i2} copies of 2, etc, and the rightmost one is a product of t-multinomial coefficients (see for instance [13, 17]). The original motivation of [14] is to bijectively prove an equality of a specialization of modified Macdonald polynomials in (1.2) via an inversion flip operator. It turns out that such operator is of central importance in deriving a compact formula for the modified Macdonald polynomials [5].

Let us recall a formula from [14] to explain the concept of Mahonian statistics. Let τ' be a filling obtained by transposing the filling τ, then the polynomial of $\mathsf{maj}(\tau')$ over $\tau\in[\sigma]$ also equals (2.1), namely,

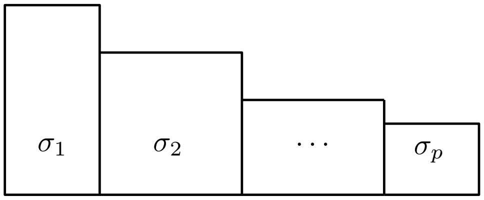
$$\sum_{\tau\in[\sigma]} t^{\mathsf{inv}(\tau)} = \sum_{\tau\in[\sigma]} t^{\mathsf{maj}(\tau')}.$$

Throughout the paper, any statistic whose distribution over $\mathcal{T}(\lambda)$ or $\mathcal{T}(\lambda')$ equals the distribution of inv on $\mathcal{T}(\lambda)$ is called a *Mahonian statistic*. Therefore, $\mathsf{quinv}, \mathsf{inv}, \mathsf{maj}$ are Mahonian statistics. Besides, we define $\chi(\mathcal{A})=1$ if the statement $\mathcal{A}$ is true; and $\chi(\mathcal{A})=0$ otherwise.

3. Reverse operator and flip operator

This section is concentrated on two operators, reverse operator and a queue inversion flip operator tailored to the statistic quinv given by Ayyer, Mandelshtam and Martin [1]. The latter was inspired by the column switch operator for the statistic inv by Loehr and Niese [14].

Both operators are related to a decomposition of the Young diagram of λ into rectangles [5]. Each Young diagram $\mathsf{dg}(\lambda)$ is regarded as a concatenation of maximal rectangles in a way that the heights of rectangles are strictly decreasing from left to right. For any $\sigma\in\mathcal{T}(\lambda)$, let σ_i be the filling of the ith rectangle of $\mathsf{dg}(\lambda)$ and $\sigma=\sigma_1\sqcup\cdots\sqcup\sigma_p$ where p is the number of rectangles of $\mathsf{dg}(\lambda)$; see Figure 3.1 for an example.



3	3	2
1	5	1
2	4	3
1	3	3

2	3	3
1	5	1
3	4	2
3	3	1

FIGURE 3.1. A decomposition of $\sigma = \sigma_1 \sqcup \sigma_2 \sqcup \cdots \sqcup \sigma_p$ (left), a filling σ of a rectangle diagram (middle) and its reverse (right).

Definition 1 (reverse operator)**.** For a partition λ and $\sigma = \sigma_1 \sqcup \cdots \sqcup \sigma_p \in \mathcal{T}(\lambda)$, define $\sigma^r = \sigma_1^r \sqcup \cdots \sqcup \sigma_p^r$ as the reverse of σ where the filling σ^r is obtained by reversing the sequence of entries of each row; see Figure 3.1.

We adopt some notations from [1, 15] to describe the flip operator. In what follows, we denote by λ_i the size of the ith row of λ, λ_i' the size of the ith column of λ, and λ' the transpose of λ. We say that i is λ-*compatible* if $\lambda_i' = \lambda_{i+1}' \geq 1$.

Definition 2. (flip operator) For $\sigma \in \mathcal{T}(\lambda)$ and λ-compatible i, let $t_i^{(r)}$ be the operator that acts on σ by interchanging the entries $\sigma(r,i)$ and $\sigma(r,i+1)$. For $1 \leq r, s \leq \lambda_i'$, let

$$t_i^{[r,s]} := t_i^{(r)} \circ t_i^{(r+1)} \cdots \circ t_i^{(s)}$$

denote the *flip operator* that swaps entries of boxes (x,i) and $(x,i+1)$ for all x with $r \leq x \leq s$. The *flip operator* ρ_i^r is defined as follows: if columns i and $i+1$ are identical in σ, then $\rho_i^r(\sigma) = \sigma$; otherwise, let k be the maximal integer such that $\sigma(k,i) \neq \sigma(k,i+1)$ and $k \leq r$. Let h be maximal such that $h \leq k$, $\sigma(h,i) \neq \sigma(h,i+1)$ and

$$\mathcal{Q}(\sigma(h,i), \sigma(h-1,i), \sigma(h-1,i+1)) = \mathcal{Q}(\sigma(h,i+1), \sigma(h-1,i), \sigma(h-1,i+1)),$$

where $\sigma(0,i) = \infty$ for all i. Define $\rho_i^r(\sigma) = t_i^{[h,k]}$, that is, reverse the pair of entries in every row between rows h and k, columns i and $i+1$. We call row k (or h) *the starting row* (or *the ending row*) of ρ_i^r. For simplicity, we denote

$$\rho_i = \rho_i^{\lambda_i'} \quad \text{and} \quad t_i = t_i^{(\lambda_i')}.$$

By definition $\rho_i^r \circ \rho_i^r(\sigma) = \sigma$, that is, ρ_i^r is an involution on $\mathcal{T}(\lambda)$.

Remark 1. We point out two differences between flip operator in Definition 2 and the queue inversion flip operator introduced in [1, 15]. First the latter always starts from the topmost row, that is ρ_i, while we allow the flip operator to begin from any row. Second, we add the condition $\sigma(h,i) \neq \sigma(h,i+1)$ to the terminating row h. These two modifications are intended to develop a precise relation between the change of quinv, inv and $\mathcal{N}\mathsf{des}$ (see Theorem 3), which contributes to the proof of Theorem 1.

Example 1. *Given a filling σ as below, $\rho_1(\sigma) = t_1^{[3,5]}(\sigma)$ is generated as follows. Since* $\boxed{9\,|\,3}$ *is the topmost row with different entries, the flipping process ρ_1 starts from this row, i.e., row*

5 and continues until row 3, where $\mathcal{Q}(3,5,8)=\mathcal{Q}(9,5,8)=1$. *Consequently* $\rho_1 = t_1^{[3,5]}$.

3	3	→	3	3	→	3	3	→	3	3
9	3		3	9		3	9		3	9
2	5		2	5		5	2		5	2
3	9		3	9		3	9		9	3
5	8		5	8		5	8		5	8
9	3		9	3		9	3		9	3

4. A roadmap for the proof of Theorem 1

The purpose of this section is to present our strategy to Theorem 1. The proof is bijective, namely, for a partition λ, we will construct a bijection $\varphi : \mathcal{T}(\lambda) \to \mathcal{T}(\lambda)$ satisfying

$$(\mathsf{quinv}, \mathsf{maj})(\varphi(\sigma)) = (\mathsf{inv}, \mathsf{maj})(\sigma). \tag{4.1}$$

In particular, if $\mathsf{dg}(\lambda)$ is a rectangle, we prove that $(\mathsf{inv}, \mathsf{quinv}, \mathsf{maj})(\varphi(\sigma)) = (\mathsf{quinv}, \mathsf{inv}, \mathsf{maj})(\sigma)$; otherwise, we find a filling σ such that (1.5) is no longer true (see Remark 2).

The bijection φ is a composition of two bijections associated with the flip operator ρ_i^k. The first one γ is described in Theorem 2 (see below), which is reduced to the reverse operator if $\mathsf{dg}(\lambda)$ is a rectangle. Let $\sigma = \sigma_1 \sqcup \cdots \sqcup \sigma_p$, define

$$\kappa(\sigma) := \sum_{i=1}^{p} (\mathsf{quinv}(\sigma_i) - \mathsf{inv}(\sigma_i^r)). \tag{4.2}$$

For any filling τ, τ_1 represents the leftmost rectangle in the decomposition of τ.

Theorem 2. *There is a bijection* $\gamma : \mathcal{T}(\lambda) \to \mathcal{T}(\lambda)$ *satisfying* $\gamma(\sigma) \sim \sigma$,

$$\mathsf{quinv}(\gamma(\sigma)) = \mathsf{inv}(\sigma) + \kappa(\gamma(\sigma)), \tag{4.3}$$

$$\mathsf{maj}(\gamma(\sigma)) = \mathsf{maj}(\sigma), \tag{4.4}$$

$$\mathcal{N}\mathsf{des}(\sigma_1) = \mathcal{N}\mathsf{des}((\gamma(\sigma)_1)^r) \tag{4.5}$$

and the topmost rows of σ *and* $\gamma(\sigma)$ *are reverse of each other.*

The second bijection $\theta : \mathcal{T}(\lambda) \to \mathcal{T}(\lambda)$ acts on each rectangle of the fillings independently and decreases the number of queue inversions by $\kappa(\gamma(\sigma))$ but preserves the major index, by which we find the bijection φ with property (4.1). Since both bijections γ, θ are constructed by the involution ϕ_i given as below, we first prove Theorem 3 (see below) in Section 5, and then establish Theorem 1 and 2 in Section 6.

Theorem 3. *For a partition* λ *and a* λ*-compatible* i, *let* $\sigma \in \mathcal{T}(\lambda)$ *and* x_i *be the number of non-descents in the* i*th column of* σ. *Then there is an involution* $\phi_i : \mathcal{T}(\lambda) \to \mathcal{T}(\lambda)$ *such that* $\phi_i(\sigma) \sim \sigma$, *and for* $\nu \in \{\mathsf{inv}, \mathsf{quinv}\}$,

$$\mathsf{maj}(\phi_i(\sigma)) = \mathsf{maj}(\sigma), \tag{4.6}$$

$$\nu(\phi_i(\sigma)) = \nu(\sigma) + x_{i+1} - x_i, \tag{4.7}$$

$$\mathcal{N}\mathsf{des}(\phi_i(\sigma)) = (i, i+1) \circ \mathcal{N}\mathsf{des}(\sigma), \tag{4.8}$$

where $(i,i+1)\circ(\ldots x_i,x_{i+1}\ldots)=(\ldots x_{i+1},x_i\ldots)$.

5. An involution on arbitrary fillings

This section contains a sequence of auxiliary Lemmas and theorems that head to Theorem 3. We begin by discussing the change of statistics quinv, inv and maj by the reverse operator and the flip operator; then proceed by exploiting these properties to confirm Theorem 3.

5.1. **Operators and statistics.** Let $\sigma|_i^j$ be the segment of σ from row i through row j. For any filling σ of a rectangle diagram, we are able to express $\kappa(\sigma)$ explicitly.

Lemma 4. *For* $\lambda=(n^m)$ *and* $\sigma\in\mathcal{T}(\lambda)$, *let* x_i *be the number of non-descents in column* i *of* σ. *Then, we have*

$$\mathsf{quinv}(\sigma)-\mathsf{inv}(\sigma^r)=\mathsf{inv}(\sigma)-\mathsf{quinv}(\sigma^r)=\sum_{i=1}^{n}x_i(n-2i+1). \tag{5.1}$$

Proof. Since (5.1) is trivially true for $m=1$ or $n=1$, we assume $m\geq 2$ and $n\geq 2$. Given $\lambda=(n^m)$ and $\sigma\in\mathcal{T}(\lambda)$, by definition for $\nu\in\{\mathsf{inv},\mathsf{quinv}\}$,

$$\nu(\sigma)=\nu(\sigma|_1^{m-1})+\nu(\sigma|_{m-1}^{m})-\nu(\sigma|_{m-1}^{m-1}), \tag{5.2}$$

which eventually leads to

$$\nu(\sigma)=\sum_{i=1}^{m-1}\nu(\sigma|_i^{i+1})-\sum_{i=2}^{m-1}\nu(\sigma|_i^{i}). \tag{5.3}$$

In view of $\mathsf{inv}(\sigma|_i^i)=\mathsf{quinv}(\sigma^r|_i^i)$, we come to

$$\mathsf{inv}(\sigma)-\mathsf{quinv}(\sigma^r)=\sum_{i=1}^{m-1}(\mathsf{inv}(\sigma|_i^{i+1})-\mathsf{quinv}(\sigma^r|_i^{i+1})), \tag{5.4}$$

by which the task of counting $\mathsf{inv}(\sigma)-\mathsf{quinv}(\sigma^r)$ is reduced to counting the ones of an $n\times 2$ rectangle, that is the case $m=2$.

Consider $\mu=(n^2)$, suppose that $\tau=\begin{array}{|c|c|c|c|}\hline a_1 & a_2 & \cdots & a_n\\\hline b_1 & b_2 & \cdots & b_n\\\hline\end{array}$ and we calculate the difference

$$\mathsf{inv}(\tau)-\mathsf{quinv}(\tau^r)=\sum_{1\leq i<j\leq n}\mathcal{Q}(a_i,b_i,a_j)+\mathcal{Q}(b_i,\infty,b_j)-\mathcal{Q}(a_j,b_j,b_i)-\mathcal{Q}(0,a_j,a_i), \tag{5.5}$$

by distinguishing whether (a_i,b_i) or (a_j,b_j) is a descent pair of τ.

(1) If $a_i>b_i$, then $\mathcal{Q}(a_i,b_i,a_j)+\mathcal{Q}(b_i,\infty,b_j)=\chi(b_i<a_j<a_i)+\chi(b_i>b_j)$; otherwise $a_i\leq b_i$ and $\mathcal{Q}(a_i,b_i,a_j)+\mathcal{Q}(b_i,\infty,b_j)=\chi(a_j<a_i)+\chi(b_i<a_j)+\chi(b_i>b_j)$.
(2) If $a_j>b_j$, then $\mathcal{Q}(a_j,b_j,b_i)+\mathcal{Q}(0,a_j,a_i)=\chi(b_j<b_i<a_j)+\chi(a_j<a_i)$; otherwise $\mathcal{Q}(a_j,b_j,b_i)+\mathcal{Q}(0,a_j,a_i)=\chi(a_j<a_i)+\chi(b_i<a_j)+\chi(b_i>b_j)$.

It is easily seen that the summand on the RHS of (5.5) equals zero if both (a_i, b_i) and (a_j, b_j) are descents or non-descents of τ. Otherwise if (a_i, b_i) is not a descent whereas (a_j, b_j) is a descent, then

$$
\begin{aligned}
&\mathcal{Q}(a_i, b_i, a_j) + \mathcal{Q}(b_i, \infty, b_j) - \mathcal{Q}(a_j, b_j, b_i) - \mathcal{Q}(0, a_j, a_i)\\
&= \chi(b_i < a_j) + \chi(b_i > b_j) - \chi(b_j < b_i < a_j)\\
&= \chi(b_j \geq b_i) + \chi(b_i > b_j) = 1
\end{aligned}
\tag{5.6}
$$

Similarly if (a_i, b_i) is a descent, but (a_j, b_j) is not a descent, we have

$$
\begin{aligned}
&\mathcal{Q}(a_i, b_i, a_j) + \mathcal{Q}(b_i, \infty, b_j) - \mathcal{Q}(a_j, b_j, b_i) - \mathcal{Q}(0, a_j, a_i)\\
&= \chi(b_i < a_j < a_i) - \chi(a_j < a_i) - \chi(b_i < a_j)\\
&= -\chi(b_i \geq a_j) - \chi(b_i < a_j) = -1
\end{aligned}
\tag{5.7}
$$

Plugging (5.6) and (5.7) back to (5.5) gives that

$$
\begin{aligned}
\mathsf{inv}(\tau) - \mathsf{quinv}(\tau^r) &= |\{(i,j) : a_i \leq b_i, a_j > b_j,\, 1 \leq i < j \leq n\}|\\
&\qquad - |\{(i,j) : a_i > b_i, a_j \leq b_j,\, 1 \leq i < j \leq n\}|,\\
&= |\{(i,j) : a_i \leq b_i,\, 1 \leq i < j \leq n\}| - |\{(i,j) : a_j \leq b_j,\, 1 \leq i < j \leq n\}|\\
&= \sum_{i=1}^{n} \chi(a_i \leq b_i)(n - 2i + 1).
\end{aligned}
$$

Therefore the second equation of (5.1) holds by combining (5.4). An equivalent form of this equation is

$$
\mathsf{quinv}(\sigma) - \mathsf{inv}(\sigma^r) = \sum_{i=1}^{n} x_i(n - 2i + 1),
$$

which is obtained by substituting x_i by x_{n-i+1} in the summand of (5.1). Thus the proof is complete. □

Let us review the central property of the flip operator.

Lemma 5. [1] *Given a partition λ and a λ-compatible i. For any $\sigma \in \mathcal{T}(\lambda)$, let a, b be entries in the topmost row of columns i and $i+1$. Then*

$$
\mathsf{maj}\,(\rho_i(\sigma)) = \mathrm{maj}(\sigma), \tag{5.8}
$$

$$
\mathsf{quinv}\,(\rho_i(\sigma)) = \mathsf{quinv}(\sigma) + \chi(a > b) - \chi(a < b). \tag{5.9}
$$

We extend Lemma 5 by analyzing the operator ρ_i^r that initiates from any row r (see Lemma 6). Though the proof of Lemma 6 is analogous to the one of Lemma 5, we include a proof to make the paper self-contained.

A significant observation is that given a filling $\begin{array}{|c|c|}\hline a & b\\ \hline c & d\\ \hline\end{array}$ of a square, $\mathcal{Q}(a, c, b) = \mathcal{Q}(a, d, b)$ if and only if $\mathcal{Q}(a, c, d) = \mathcal{Q}(b, c, d)$. That is, both (a, c, b) and (a, d, b) are inversion triples or neither is an inversion triple, if and only if both (a, c, d) and (b, c, d) are queue inversion triples or neither is a queue inversion triple.

Lemma 6. *For a partition λ and a λ-compatible i, let $\sigma \in \mathcal{T}(\lambda)$ and suppose that $\rho_i^r = t_i^{[\kappa_1,\kappa_2]}$. Let $\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}$ and $\begin{array}{|c|c|}\hline s & t \\ \hline u & v \\ \hline\end{array}$ be parts of σ such that $\begin{array}{|c|c|}\hline c & d \\ \hline\end{array}$ is the starting row κ_2 and $\begin{array}{|c|c|}\hline s & t \\ \hline\end{array}$ is the ending row κ_1. Set $\sigma(0,i) = \infty$ and $\sigma(\lambda_i'+1,i) = 0$ for all i. If $\mathcal{Q}(a,c,d) = \mathcal{Q}(b,c,d) = 0$, then*

$$\mathsf{quinv}(\sigma) + 1 = \mathsf{quinv}(\rho_i^r(\sigma)). \tag{5.10}$$

Equivalently, if $\mathcal{Q}(a,c,d) = \mathcal{Q}(b,c,d) = 1$, then

$$\mathsf{quinv}(\sigma) - 1 = \mathsf{quinv}(\rho_i^r(\sigma)). \tag{5.11}$$

If $\mathcal{Q}(s,u,t) = \mathcal{Q}(s,v,t) = 0$, then

$$\mathsf{inv}(\sigma) + 1 = \mathsf{inv}(\rho_i^r(\sigma)). \tag{5.12}$$

Equivalently, if $\mathcal{Q}(s,u,t) = \mathcal{Q}(s,v,t) = 1$, then

$$\mathsf{inv}(\sigma) - 1 = \mathsf{inv}(\rho_i^r(\sigma)). \tag{5.13}$$

For all cases, i.e., $\mathcal{Q}(a,c,d) = \mathcal{Q}(b,c,d)$ or $\mathcal{Q}(s,u,t) = \mathcal{Q}(s,v,t)$, we have

$$\mathsf{maj}(\sigma) = \mathsf{maj}(\rho_i^r(\sigma)). \tag{5.14}$$

Proof. Since the proofs of (5.12), (5.13) and (5.14) for $\mathcal{Q}(s,u,t) = \mathcal{Q}(s,v,t)$ are exactly the same as the ones of others, we only prove the latter. We first identify the equivalence between (5.10) and (5.11). Let $\rho_i^r(\sigma) = \tau$, then $\sigma = \rho_i^r(\tau)$ because ρ_i^r is an involution. No matter $c < d$ or $c > d$, $\mathcal{Q}(a,c,d) = \mathcal{Q}(b,c,d) = 0$ in σ if and only if $\mathcal{Q}(a,d,c) = \mathcal{Q}(b,d,c) = 1$ in τ. It follows that (5.11) holds for τ, that is, $\mathsf{quinv}(\tau) - 1 = \mathsf{quinv}(\rho_i^r(\tau))$, if and only if (5.10) is true for σ. So we only need to prove (5.10).

Suppose that $\rho_i^r = t_i^{[\kappa_1,\kappa_2]}$, the only possible change of maj may occur on the borders, that is between rows $\kappa_2 + 1$ and κ_2 and between rows $\kappa_1 - 1$ and κ_1. Here we examine the former one in that the latter one is discussed in a similar manner.

Let $\pi = \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}$, then $\rho_i^r(\pi) = \begin{array}{|c|c|}\hline a & b \\ \hline d & c \\ \hline\end{array}$ and we shall verify that $\mathsf{maj}(\pi) = \mathsf{maj}(\rho_i^r(\pi))$. If $\mathcal{Q}(a,c,d) = \mathcal{Q}(b,c,d) = 0$, then $c < x \le d$ for $x \in \{a,b\}$ if $c < d$; and $d < c < x$ or $x \le d < c$ for $x \in \{a,b\}$ if $c > d$. If $\mathcal{Q}(a,c,d) = \mathcal{Q}(b,c,d) = 1$, then $x \le c < d$ or $c < d < x$ for $x \in \{a,b\}$ if $c < d$; and $d < x \le c$ for $x \in \{a,b\}$ if $c > d$. For all cases, we always have $\mathsf{maj}(\pi) = \mathsf{maj}(\rho_i^r(\pi))$, which further leads to (5.14).

It remains to prove (5.10). First ρ_i^r preserves the number of queue inversion triples induced between columns i and $i+1$ other than the starting row r. This is true because of the following argument.

Let $\pi = \begin{array}{|c|c|}\hline \alpha & \beta \\ \hline \gamma & \delta \\ \hline\end{array}$, then if both rows of π are reversed by ρ_i^r, we must have $\delta \neq \gamma$, $\mathcal{Q}(\alpha,\gamma,\delta) \neq \mathcal{Q}(\beta,\gamma,\delta)$ and $\mathcal{Q}(\beta,\gamma,\delta) \neq \mathcal{Q}(\beta,\delta,\gamma)$, thus $\mathcal{Q}(\alpha,\gamma,\delta) = \mathcal{Q}(\beta,\delta,\gamma)$; otherwise if $\begin{array}{|c|c|}\hline \alpha & \beta \\ \hline\end{array}$ is the terminating row k, we get $\mathcal{Q}(\alpha,\gamma,\delta) = \mathcal{Q}(\beta,\gamma,\delta)$ by definition. In all above cases, the number of queue inversion triples between column i and $i+1$ is maintained by ρ_i^r and we are left with the case that $\begin{array}{|c|c|}\hline \gamma & \delta \\ \hline\end{array} = \begin{array}{|c|c|}\hline c & d \\ \hline\end{array}$ is the starting row r, that is, $\begin{array}{|c|c|}\hline \alpha & \beta \\ \hline\end{array} = \begin{array}{|c|c|}\hline a & b \\ \hline\end{array}$. Under the conditions

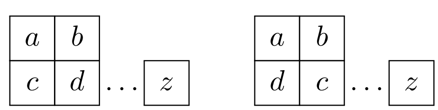
	$(\mathcal{Q}(a,c,z),\mathcal{Q}(b,d,z))$	$(\mathcal{Q}(a,d,z),\mathcal{Q}(b,c,z))$
$z > d \geq a > b > c$	$(0,1)$	$(1,0)$
$z > d \geq b \geq a > c$	$(0,1)$	$(1,0)$
$d \geq z \geq a > b > c$	$(0,0)$	$(0,0)$
$d \geq z \geq b \geq a > c$	$(0,0)$	$(0,0)$
$d \geq a > z \geq b > c$	$(1,0)$	$(1,0)$
$d \geq b > z \geq a > c$	$(0,1)$	$(0,1)$
$d \geq a > b > z > c$	$(1,1)$	$(1,1)$
$d \geq b \geq a > z > c$	$(1,1)$	$(1,1)$
$d \geq a > b > c \geq z$	$(0,1)$	$(1,0)$
$d \geq b \geq a > c \geq z$	$(0,1)$	$(1,0)$

TABLE 1. The change of queue inversion triples for the cases $d \geq a > b > c$ or $d \geq b \geq a > c$.

that $\mathcal{Q}(a,c,d)=0$ and $c \neq d$, we conclude that $\mathcal{Q}(a,d,c)=1$, thus yielding that

$$\mathsf{quinv}(\pi)+1=\mathsf{quinv}(\rho_i^r(\pi)). \tag{5.15}$$

Second we claim that the number of queue inversion triples induced by column i or $i+1$ and column j for all $j > i+1$ is invariant under ρ_i^r, namely, invariant under $t_i^{(r)}$ and $t_i^{(k)}$. Here we only prove the claim for $t_i^{(r)}$ and omit the other analogous case. Consider the square of entries a,b,c,d and let $z=\sigma(r,j)$, as shown below.



We look at all possible total orders of a,b,c,d, namely, $c > d \geq a > b$, $c > d > b \geq a$, $b > c > d \geq a$, $d \geq a > b > c$, $d \geq b \geq a > c$, $a > c > d \geq b$, $a > b > c > d$, $b \geq a > c > d$. For each of these scenarios, there are five possible ways to insert z, as outlined in Table 1. Note that only cases for $d \geq a > b > c$ or $d \geq b \geq a > c$ are listed because other ones follow in the same manner. One readily sees that $\mathcal{Q}(a,c,z)+\mathcal{Q}(b,d,z)=\mathcal{Q}(a,d,z)+\mathcal{Q}(b,c,z)$, that is, the number of queue inversion numbers is preserved for this case, which together with (5.15) proves (5.10).

□

5.2. **The proof of Theorem 3.** First the fillings of a 2×2 block of squares are classified into two kinds, descent block and neutral block, by which each filling of a $2\times n$ rectangle is considered as a sequence of overlapping blocks.

For any filling τ of a 2×2 square, if exactly one of the columns of τ forms a descent, we call it a *right descent block* or *left descent block* depending on whether the right or the left column forms a descent pair. Otherwise both columns of τ form descent or non-descent pairs, we name it a *neutral block*.

As a consequence, every filling σ of a $2\times n$ rectangle is uniquely decomposed into a sequence of neutral blocks (could be empty) and descent blocks from top to bottom. Two descent blocks are *neighbours* if there are only neutral blocks (could be empty) between them and they are *compatible* if both are left or right descent blocks. All descent blocks are further classified into three types.

Definition 3. Let $\tau = \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}$ be a descent block, define

- $\tau\in\mathcal{A}$ if $c\neq d$ and $\{c,d\}$ is the set of the smallest and the largest integers of a,b,c,d, that is, $d\geq b\geq a>c$, $d\geq a>b>c$, $c\geq b\geq a>d$ or $c\geq a>b>d$.
- $\tau\in\mathcal{B}$ if $a\neq b$ and $\{a,b\}$ is the set of the smallest and the largest integers of a,b,c,d, that is, $a>c\geq d\geq b$, $a>d>c\geq b$, $b>c\geq d\geq a$ or $b>d>c\geq a$.
- $\tau\in\mathcal{C}$ if $a>d\geq b>c$, $d\geq a>c\geq b$, $b>c\geq a>d$, $c\geq b>d\geq a$.

For $\tau\in\mathcal{A}\cup\mathcal{B}$, we have $\mathcal{Q}(a,c,d)=\mathcal{Q}(b,c,d)$ while for $\tau\in\mathcal{C}$, $\mathcal{Q}(a,c,d)\neq\mathcal{Q}(b,c,d)$. We are going to define a map ε that sends a descent block to a flip operator as follows:

For any partition λ and a λ-compatible i, let $\sigma\in\mathcal{T}(\lambda)$ and $\tau = \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}$ be a descent block of σ between column i and $i+1$. Suppose that τ is a block from row r to row $r+1$. If $\tau\in\mathcal{A}$, set $\varepsilon(\tau)=\rho_i^r$; otherwise define $\varepsilon(\tau)=\rho_i^\kappa$ where κ is chosen as follows: take the smallest κ such that $\kappa>r$ and

$$\mathcal{Q}(\sigma(\kappa+1,i),\sigma(\kappa,i),\sigma(\kappa,i+1))=\mathcal{Q}(\sigma(\kappa+1,i+1),\sigma(\kappa,i),\sigma(\kappa,i+1))=\chi(a\leq c). \quad (5.16)$$

An important point to note here is that we have to validate the existence of κ for $\tau\notin\mathcal{A}$, so that $\varepsilon(\tau)$ is well-defined. We say that a block ρ is *above* τ if the top row of ρ is above the top row of τ, emphasizing that ρ and τ could overlap at most one row.

Lemma 7. *Let* $\tau = \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}$ *be a descent block, then $\varepsilon(\tau)$ is well-defined if (i) all blocks above τ are neutral or (ii) the neighbour descent block of τ from above belongs to $\mathcal{B}$ or is compatible with τ.*

(1) For (i), let $\varepsilon(\tau)=\rho_i^\kappa$ and $\tau\notin\mathcal{A}$, then for $r<j\leq\kappa$, we have $\sigma(j,i)>\sigma(j,i+1)$ if $a>c$; and $\sigma(j,i)<\sigma(j,i+1)$ otherwise.

(2) For (ii), let ρ denote such neighbour of τ, then the terminating row of $\varepsilon(\rho)$ is above the starting row of $\varepsilon(\tau)$.

Proof. Without loss of generality, assume $a>c$, $b\leq d$. For (i), consider $\tau\in\mathcal{B}\cup\mathcal{C}$ and κ does not exist for $\varepsilon(\tau)$, then exactly one of $\mathcal{Q}(\sigma(j,i),\sigma(j-1,i),\sigma(j-1,i+1))$ and $\mathcal{Q}(\sigma(j,i+1),\sigma(j-1,i),\sigma(j-1,i+1))$ equals one for all $j>r$. In this case, we shall see that $\sigma(j,i)>\sigma(j,i+1)$ for all $j>r$ by induction on j.

First it is true if $j = r+1$, that is, $a > b$ because if $a \leq b$, then $c < a \leq b \leq d$, contradicting the condition that $\tau \notin \mathcal{A}$. Suppose that $\sigma(x, i) > \sigma(x, i+1)$ for all $r < x \leq j-1$, we are going to prove that it is also true for $x = j$.

Recall that $\mathcal{Q}(\sigma(j,i), \sigma(j-1,i), \sigma(j-1,i+1)) \neq \mathcal{Q}(\sigma(j,i+1), \sigma(j-1,i), \sigma(j-1,i+1))$. Combined with the condition that the block $\sigma(j,i)$, $\sigma(j,i+1)$, $\sigma(j-1,i)$ and $\sigma(j-1,i+1)$ is a neutral block and $\sigma(j-1,i) > \sigma(j-1,i+1)$, we conclude that $\sigma(j,i) > \sigma(j,i+1)$, completing the proof of the assertion.

Take $j = \lambda_i'$, thus leading to $\sigma(\lambda_i', i) > \sigma(\lambda_i', i+1)$. However, $\mathcal{Q}(0, \sigma(\lambda_i', i), \sigma(\lambda_i', i+1)) = 0$, implying that $\kappa = \lambda_i'$, contradicting the non-existence of κ. In consequence, κ exists and $\varepsilon(\tau)$ is well-defined for (i) with $a > c$. Similarly one can prove the statement for $a \leq c$ and in this case $\sigma(j,i) < \sigma(j,i+1)$ for $r < j \leq \kappa$. This finishes the proof of (1).

For (ii), suppose that $\rho = \begin{array}{|c|c|}\hline \alpha & \beta \\ \hline \gamma & \delta \\ \hline \end{array}$ where $\alpha > \gamma$ is compatible with τ. Let c and γ have coordinates (r,i) and (j,i) respectively, where $j \geq r+1$.

Let row k_1 be the terminating row for $\varepsilon(\rho)$, and row k_2 be the starting row for $\varepsilon(\tau)$, we will show that k_2 exists and $k_1 > k_2$. Suppose that k_2 does not exist, then $\gamma > \delta$ by (1) and consequently $\mathcal{Q}(\alpha, \gamma, \delta) = \mathcal{Q}(\beta, \gamma, \delta) = 0$ by noting $\alpha > \gamma$ and $\beta \leq \delta$. This is against the non-existence of k_2, thus k_2 exists and $\varepsilon(\tau)$ is well-defined. We will discuss the type of τ to prove $k_1 > k_2$. If $\tau \in \mathcal{A}$, then $k_2 = r$ and $k_1 \geq r+1$ as $\mathcal{Q}(a,c,d) = \mathcal{Q}(b,c,d)$, thus $k_1 > k_2$. If $\tau \in \mathcal{B} \cup \mathcal{C}$, we assert that $k_2 \leq j$. Suppose that $k_2 > j$, then (1) guarantees that $\gamma > \delta$. Therefore $\rho \in \mathcal{B}$ and $\mathcal{Q}(\alpha, \gamma, \delta) = \mathcal{Q}(\beta, \gamma, \delta) = 0$, resulting in $k_2 = j$, a contradiction. As a result, $k_2 \leq j$, which further leads to $k_2 < k_1$ by definition of k_1 and the minimality of k_2. So we conclude that $\varepsilon(\tau)$ is well-defined for (ii) and (2) is true when ρ and τ are compatible.

Suppose that ρ and τ are incompatible, and $\rho \in \mathcal{B}$. Without loss of generality, we assume that τ is a left descent block and ρ is a right descent block. Since the terminating row of $\varepsilon(\rho)$ is above row j, that is row $\begin{array}{|c|c|}\hline \gamma & \delta \\ \hline \end{array}$, it suffices to show that the starting row of $\varepsilon(\tau)$ is below or equal to row j. If it is not below row j, we shall see that it must be row j. Recall that (1) says that $\gamma > \delta$. In combination of $\alpha \leq \gamma$, $\beta > \delta$ and $\rho \in \mathcal{B}$, it follows that $\mathcal{Q}(\alpha, \gamma, \delta) = \mathcal{Q}(\beta, \gamma, \delta) = 0$, yielding that the starting row of $\varepsilon(\tau)$ must be row j, that is, $k_1 \geq j+1 > k_2 = j$, as desired.

□

We are in a position to make significant use of the operators $\varepsilon(\tau)$ to establish Theorem 3.

Proof of Theorem 3. Let σ^i be σ restricted to column i. For a given σ, the filling $\sigma^i \cup \sigma^{i+1}$ is decomposed into a sequence of decent blocks and neutral blocks. The map ϕ_i is defined as follows. For the trivial case that there is no descent block, that is, $\mathsf{ndes}(\sigma^{i+1}) = \mathsf{ndes}(\sigma^i)$, thus take $\phi_i(\sigma) = \sigma$ which clearly satisfies (4.6) to (4.8).

First find all j such that $\mathcal{Q}(\sigma(j,i), \sigma(j-1,i), \sigma(j-1,i+1)) = \mathcal{Q}(\sigma(j,i+1), \sigma(j-1,i), \sigma(j-1,i+1))$ and divide $\sigma^i \cup \sigma^{i+1}$ into a sequence of components by all these j. Let $\mathcal{S}_m$ be the mth component from top to bottom for $1 \leq m \leq k$, then $\sigma^i \cup \sigma^{i+1} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_k$.

For any integer m with $1 \leq m \leq k$, suppose that $\mathcal{S}_m$ is the segment of $\sigma^i \cup \sigma^{i+1}$ from row κ_1 through row κ_2 with $\kappa_1 \leq \kappa_2$. Consider two blocks on the borders of $\mathcal{S}_m$, let τ_2 (resp. τ_1) be the block of $\sigma_i \cup \sigma_{i+1}$ between rows κ_2 and $\kappa_2 + 1$ (resp. between rows $\kappa_1 - 1$ and κ_1). We denote by $\mathcal{T}_m$ the set of descent blocks from row $\kappa_2 + \chi(\tau_2 \in \mathcal{A})$ through row $\kappa_1 - \chi(\tau_1 \in \mathcal{B})$, in which π_m is the topmost descent block, thus $\pi_m = \tau_2$ if $\tau_2 \in \mathcal{A}$.

For $1 \leq m \leq k$, if $|\mathcal{T}_m|$ is odd, we apply $t_i^{[\kappa_1,\kappa_2]}$ on the filling σ. Denote the resulting filling by $\phi_i(\sigma)$ and we shall prove that ϕ_i is an involution with the properties (4.6) to (4.8).

First we verify that $\varepsilon(\pi_m) = t_i^{[\kappa_1,\kappa_2]}$. If $\pi_m \in \mathcal{A}$, then clearly by definition $\varepsilon(\pi_m) = t_i^{[\kappa_1,\kappa_2]}$. Otherwise $\pi_m \in \mathcal{B} \cup \mathcal{C}$, the terminating row of $\varepsilon(\pi_m)$ must be row κ_1 and it suffices to prove (5.16) for $\kappa = \kappa_2$. In this case, $\tau_2 \notin \mathcal{A}$, thus τ_2 is neutral or $\tau_2 \in \mathcal{B}$ is the neighbour of π_m from above. If τ_2 is neutral, then (1) of Lemma 7 says that $\sigma(\kappa_2, i) > \sigma(\kappa_2, i+1)$, thus confirming that $\mathcal{Q}(\sigma(\kappa_2 + 1, i), \sigma(\kappa_2, i), \sigma(\kappa_2, i + 1)) = \mathcal{Q}(\sigma(\kappa_2 + 1, i + 1), \sigma(\kappa_2, i), \sigma(\kappa_2, i + 1)) = 0$. Otherwise $\tau_2 \in \mathcal{B}$ is the neighbour of π_m from above. It follows from (2) of Lemma 7 that the starting row of $\varepsilon(\pi_m)$ is below the terminating row $\kappa_2 + 1$ of $\varepsilon(\tau_2)$, which implies that the starting row of $\varepsilon(\pi_m)$ must be row κ_2. In consequence, $\varepsilon(\pi_m) = t_i^{[\kappa_1,\kappa_2]}$. Second notice that ϕ_i is a product of commuting $\varepsilon(\pi_m)$, because all components $\mathcal{S}_j$ are disjoint, thus each row of $\sigma^i \cup \sigma^{i+1}$ is either fixed or reversed exactly once by ϕ_i.

We next show that ϕ_i is an involution. Let $\tau = \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline \end{array}$ be part of $\sigma^i \cup \sigma^{i+1}$ on the border between two components, that is, $\begin{array}{|c|c|}\hline a & b \\ \hline \end{array}$ is the bottom row of a component and $\begin{array}{|c|c|}\hline c & d \\ \hline \end{array}$ is the top row of its neighbour component. By construction $\mathcal{Q}(a, c, d) = \mathcal{Q}(b, c, d)$, which is preserved by ϕ_i if row $\begin{array}{|c|c|}\hline c & d \\ \hline \end{array}$ is fixed by ϕ_i; otherwise if row $\begin{array}{|c|c|}\hline c & d \\ \hline \end{array}$ is reversed by ϕ_i, then it is not hard to see that $\mathcal{Q}(a, d, c) = \mathcal{Q}(b, d, c)$, meaning that the way to divide $\sigma^i \cup \sigma^{i+1}$ into components is the same before and after ϕ_i. Further, the type of a descent block is maintained by ϕ_i, that is, if τ is a descent block, then $\tau \in \mathcal{X}$ if and only if $\phi_i(\tau) \in \mathcal{X}$ for $\mathcal{X} \in \{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$, which follows immediately from Definition 3 and the construction of ϕ_i. This shows that κ_1, κ_2 and $\mathcal{T}_m$ are invariant under ϕ_i. It follows that ϕ_i is an involution.

Finally we are going to establish (4.6) to (4.8). The equation (4.6) is a direct consequence of (5.14). For $1 \leq m \leq k$, let $\varepsilon(\pi_m)(\sigma) = \pi$. We assert that if $\varepsilon(\pi_m)$ is applied on σ in ϕ_i, then $\mathsf{quinv}(\pi) = \mathsf{quinv}(\sigma) + \mathsf{ndes}(\pi^i) - \mathsf{ndes}(\sigma^i)$; otherwise $\mathsf{ndes}(\pi^i) = \mathsf{ndes}(\sigma^i)$. The descent blocks of $\mathcal{T}_m$ are alternating in the sense that every two neighbours are incompatible. This is justified by (2) of Lemma 7, namely, $\varepsilon(\pi_m)$ terminates between two compatible neighbours. If $|\mathcal{T}_m|$ is odd, $\varepsilon(\pi_m)$ is applied on σ and consequently every left (or right) descent block of $\mathcal{T}_m$ becomes a right (or left) descent block after $\varepsilon(\pi_m)$. This can be easily checked by discussing its descent block type, for which we omit the details. In other words, let $\mathcal{R}$ (or $\mathcal{L}$) represent a right (or left) descent block, then the sequence of descent blocks $(\ldots, \mathcal{R}, \mathcal{L}, \mathcal{R}, \mathcal{L}, \ldots)$ from $\mathcal{T}_m$ becomes $(\ldots, \mathcal{L}, \mathcal{R}, \mathcal{L}, \mathcal{R}, \ldots)$ after $\varepsilon(\pi_m)$. Consequently the number of non-descents in column i is increased (or decreased) by one if and only if π_m is a left (or right) descent block, if and only if the number of queue inversion triples is increased (or decreased) by one due to Lemma 6. Equivalently,

$$\begin{aligned} \mathsf{quinv}(\pi) - \mathsf{quinv}(\sigma) &= \chi(\pi_m \text{ is a left descent}) - \chi(\pi_m \text{ is a right descent}) \\ &= \mathsf{ndes}(\pi^i) - \mathsf{ndes}(\sigma^i). \end{aligned} \tag{5.17}$$

If $|\mathcal{T}_m|$ is even, then there are an equal number of left and right descent blocks of $\mathcal{T}_m$. In other words, $\mathsf{ndes}(\pi^i) = \mathsf{ndes}(\sigma^i)$. This assertion produces that

$$\mathsf{quinv}(\phi_i(\sigma)) - \mathsf{quinv}(\sigma) = \mathsf{ndes}(\phi_i(\sigma)^i) - \mathsf{ndes}(\sigma^i) = x_{i+1} - x_i,$$

and the second equality gives that $\mathsf{ndes}(\phi_i(\sigma)^i) = x_{i+1}$, thus $\mathsf{ndes}(\phi_i(\sigma)^{i+1}) = x_i$ for that the number of descents of σ is preserved by ϕ_i. This explains (4.7) for $\nu = \mathsf{quinv}$ and (4.8). What remains is to prove (4.7) for $\nu = \mathsf{inv}$ and by the second equality of (5.17) it reduces to verify that

$$\mathsf{inv}(\pi) - \mathsf{inv}(\sigma) = \chi(\pi_m \text{ is a left descent}) - \chi(\pi_m \text{ is a right descent}) \tag{5.18}$$

for odd $|\mathcal{T}_m|$. Let α_m is the bottommost descent block of $\mathcal{T}_m$. Since $|\mathcal{T}_m|$ is odd, i.e., π_m and α_m are compatible, (5.18) becomes

$$\mathsf{inv}(\pi) - \mathsf{inv}(\sigma) = \chi(\alpha_m \text{ is a left descent}) - \chi(\alpha_m \text{ is a right descent}). \tag{5.19}$$

Notice that $\alpha_m \in \mathcal{B} \cup \mathcal{C}$ because otherwise $\alpha_m \in \mathcal{A}$ and $\mathcal{T}_m$ would not have α_m. If $\alpha_m \in \mathcal{B}$, i.e., $\tau_1 \in \mathcal{B}$, then Lemma 6 claims (5.19). If $\alpha_m \in \mathcal{C}$, we assert that

$$\sigma(\kappa_1, i) < \sigma(\kappa_1, i+1) \tag{5.20}$$

if α_m is a left descent; otherwise

$$\sigma(\kappa_1, i) > \sigma(\kappa_1, i+1). \tag{5.21}$$

This can be derived in the same manner as (1) of Lemma 7, so we omit details. On the other hand, $\tau_1 \notin \mathcal{C}$ since otherwise row κ_1 would not be a terminating row. It follows from (5.20) and (5.21) that for all cases that τ_1 is neutral or $\tau_1 \in \mathcal{A}$, (5.19) must be true, by which (4.7) for $\nu = \mathsf{inv}$ is concluded. This finishes the proof. □

Example 2. *Consider the filling σ shown as below, $\mathcal{N}\mathsf{des}(\sigma) = (x_1, x_2) = (1, 2)$ and we apply the involution ϕ_1 on these two columns as follows:*

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 5 & 4 \\ \hline 3 & 7 \\ \hline \end{array} \to \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 5 & 4 \\ \hline 7 & 3 \\ \hline \end{array}$$

First divide σ into three components $\mathcal{S}_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}$, $\mathcal{S}_2 = \begin{array}{|c|c|}\hline 5 & 4 \\ \hline \end{array}$ and $\mathcal{S}_3 = \begin{array}{|c|c|}\hline 3 & 7 \\ \hline \end{array}$ as $\mathcal{Q}(1, 5, 4) = \mathcal{Q}(2, 5, 4)$ and $\mathcal{Q}(5, 3, 7) = \mathcal{Q}(4, 3, 7)$. Then consider $\mathcal{S}_1$ and $\mathcal{S}_2$. Since $\mathcal{T}_1$ and $\mathcal{T}_2$ are empty, we do nothing.

For $\mathcal{S}_3$, $\mathcal{T}_3$ contains only one descent block $\pi_3 = \begin{array}{|c|c|}\hline 5 & 4 \\ \hline 3 & 7 \\ \hline \end{array}$. We apply $\varepsilon(\pi_3) = t_1^1$ on σ, that is, reverse the row $\begin{array}{|c|c|}\hline 3 & 7 \\ \hline \end{array}$, and obtain the right one of the picture. To sum up, $\phi_1 = \varepsilon(\pi_3)$ such that $\mathsf{maj}(\phi_1(\sigma)) = \mathsf{maj}(\sigma) = 2$, $\nu(\phi_1(\sigma)) = \nu(\sigma) + 1$ for $\nu \in \{\mathsf{inv}, \mathsf{quinv}\}$ and $\mathcal{N}\mathsf{des}(\phi_1(\sigma)) = (2, 1)$.

6. The proof of Theorems 1 and 2

The main purpose of section is to complete the proof of Theorem 1. First we present Theorem 8 which introduces the second bijection θ towards the desired bijection φ for Theorem 1. Second we discuss how Theorem 2 and 8 lead to Theorem 1 and finally we confirm Theorem 2.

Theorem 8 (second part of Theorem 1)**.** *If the diagram of λ is a rectangle, then there is a bijection $\theta : \mathcal{T}(\lambda) \to \mathcal{T}(\lambda)$ such that* $(\mathsf{inv}, \mathsf{quinv}, \mathsf{maj})\sigma = (\mathsf{quinv}, \mathsf{inv}, \mathsf{maj})\theta(\sigma)$.

Proof. For $\lambda = (n^m)$ and $\sigma \in \mathcal{T}(\lambda)$, let x_i be the number of non-descents in column i. Then $\mathcal{N}\mathsf{des}(\sigma^r) = (x_n, \dots, x_1)$. Define a bijection $\theta : \mathcal{T}(\lambda) \to \mathcal{T}(\lambda)$ as a product of ϕ_i's as follows. In the first step, apply the bijections ϕ_i for i from $n-1$ to 1 on σ^r, let $\tau_1 = \phi_1 \circ \cdots \circ \phi_{n-1}(\sigma^r)$. Theorem 3 ensures that the number of queue inversion triples is increased by

$$\nu(\tau_1) - \nu(\sigma^r) = \sum_{i=2}^{n} (x_1 - x_i) = (n-1)x_1 - \sum_{i=2}^{n} x_i$$

for $\nu \in \{\mathsf{inv}, \mathsf{quinv}\}$ and $\mathcal{N}\mathsf{des}(\tau_1) = (x_1, x_n, \dots, x_2)$. In the next step, we apply the bijections ϕ_i for i from $n-1$ to 2 on τ_1 and let $\tau_2 = \phi_2 \circ \cdots \circ \phi_{n-1}(\tau_1)$, yielding that the number of queue inversion triples is increased by $(n-2)x_2 - (x_3 + \cdots + x_n)$ and $\mathcal{N}\mathsf{des}(\tau_2) = (x_1, x_2, x_n \dots, x_3)$. Continue this process until the sequence $\mathcal{N}\mathsf{des}$ of the image becomes $(x_1, x_2, \dots, x_n)$. Denote the resulting filling by $\theta(\sigma)$ and one sees that $\mathsf{maj}(\theta(\sigma)) = \mathsf{maj}(\sigma)$ by (4.6). Further,

$$\begin{aligned} \nu(\theta(\sigma)) &= \nu(\sigma^r) + \sum_{i=1}^{n-1} (n-i)x_i - \sum_{i=2}^{n} (i-1)x_i \\ &= \nu(\sigma^r) + \sum_{i=1}^{n} x_i(n - 2i + 1). \end{aligned}$$

Compare with (5.1), we conclude that $(\mathsf{inv}, \mathsf{quinv})(\sigma) = (\mathsf{quinv}, \mathsf{inv})(\theta(\sigma))$, as claimed. ☐

Remark 2. In the table below, we present a row-equivalent filling class $[\sigma]$ of a non-rectangular diagram to disprove (1.5) for arbitrary fillings.

$[\sigma]$	3 4 1 2 3 3 3	3 4 2 1 3 3 3	3 1 2 4 3 3 3	3 1 4 2 3 3 3	3 2 4 1 3 3 3	3 2 1 4 3 3 3
maj	2	2	2	2	2	2
inv	0	1	2	1	2	3
quinv	3	2	2	1	0	1

We proceed by establishing Theorem 1 with the help of Theorem 2 and 8.

Proof of Theorem 1. For any $\sigma \in \mathcal{T}(\lambda)$, let $\tau = \gamma(\sigma)$, consider the rectangle decomposition of $\tau = \tau_1 \sqcup \cdots \sqcup \tau_p$, let $\pi = \theta(\tau_1^r) \sqcup \cdots \sqcup \theta(\tau_p^r)$ and we shall see that $\varphi(\sigma) = \pi$ satisfying $\mathsf{quinv}(\pi) = \mathsf{inv}(\sigma)$ and $\mathsf{maj}(\pi) = \mathsf{maj}(\sigma)$. Theorem 8 assures that

$$\mathsf{quinv}(\pi) - \mathsf{quinv}(\tau) = \sum_{i=1}^{p} (\mathsf{quinv}(\theta(\tau_i^r)) - \mathsf{quinv}(\tau_i)) = \sum_{i=1}^{p} (\mathsf{inv}(\tau_i^r) - \mathsf{quinv}(\tau_i)) = -\kappa(\tau).$$

In combination of $\mathsf{quinv}(\tau) - \mathsf{inv}(\sigma) = \kappa(\tau)$ by Theorem 2, we conclude that $\mathsf{quinv}(\pi) = \mathsf{inv}(\sigma)$. Furthermore, $\mathsf{maj}(\pi) = \mathsf{maj}(\sigma)$ follows directly by Theorem 2 and 8, as wished.

	$(\mathcal{Q}(a,c,z),\mathcal{Q}(b,d,z))$	$(\mathcal{Q}(b,c,z),\mathcal{Q}(a,d,z))$
$c \geq b > d > a > z$	$(1,0)$	$(1,1)$
$c \geq b > d \geq z \geq a$	$(0,0)$	$(1,0)$
$c \geq b > z > d > a$	$(0,1)$	$(1,1)$
$c \geq z \geq b > d > a$	$(0,0)$	$(0,1)$
$z > c \geq b > d > a$	$(1,0)$	$(1,1)$

TABLE 2. The change of queue inversion triples for the case $c \geq b > d > a$.

□

We focus on proving Theorem 2. The lemma below treats special cases that a is the unique smallest one among the entries a, b, c, d of a square and it plays an essential role in deriving (4.3) of Theorem 2.

Lemma 9. *For a partition $\lambda = (n^m)$, let $\sigma \in \mathcal{T}(\lambda)$ and let a, b, c, d be $\sigma(m,i)$, $\sigma(m,i+1)$, $\sigma(m-1,i)$ and $\sigma(m-1,i+1)$, respectively, where $\sigma(0,i) = \infty$ and $\sigma(m+1,i) = 0$. If $a < d < b \leq c$, we have*

$$\mathsf{quinv}(\sigma) + (n-i-1) = \mathsf{quinv}(t_i(\sigma)) \quad \textit{and} \quad \mathsf{maj}(\sigma) - 1 = \mathsf{maj}(t_i(\sigma)).$$

If $a < c < b \leq d$, then

$$\mathsf{quinv}(\sigma) - (n-i+1) = \mathsf{quinv}(t_i(\sigma)) \quad \textit{and} \quad \mathsf{maj}(\sigma) + 1 = \mathsf{maj}(t_i(\sigma)).$$

Proof. For $c \geq b > d > a$, it is clear that the major index is reduced by one after switching a and b. Further, we have $\mathcal{Q}(0,a,b) = \mathcal{Q}(b,c,d) = 1$ whereas $\mathcal{Q}(0,b,a) = \mathcal{Q}(a,c,d) = 0$, which implies that the number of queue inversion triples between columns i and $i+1$ is invariant after the interchange of a and b. Second we consider the number of queue inversion triples induced by column i or $i+1$ and column j for all $j > i+1$. There are five possible ways to insert z, as outlined in Table 2. Evidently $\mathcal{Q}(a,c,z) + \mathcal{Q}(b,d,z) + 1 = \mathcal{Q}(a,d,z) + \mathcal{Q}(b,c,z)$ for all cases. Since z is any element in row $m-1$ and column j for $i+1 < j \leq n$, the change from $\boxed{a}\boxed{b}$ to $\boxed{b}\boxed{a}$ increases the number of queue inversion triples by $n-i-1$, as desired. The proof for the case $d \geq b > c > a$ is similar, so we omit the details. □

We are now ready to prove Theorem 2. In Example 3, we illustrate how the bijections γ and φ are implemented on a given filling.

Proof of Theorem 2. We shall prove the statement by induction on the number of rows. The base case is that λ has only one row, we simply take $\gamma(\sigma) = \sigma^r$ and $\kappa(\sigma^r) = 0$. Consider λ with at least two rows, let μ be the partition obtained by removing the top row of λ. Suppose that $\gamma : \mathcal{T}(\mu) \to \mathcal{T}(\mu)$ is a bijection under which $\gamma(\tau)$ has the desired properties (4.3)–(4.5) for any $\tau \in \mathcal{T}(\mu)$. For $\sigma \in \mathcal{T}(\lambda)$, let the topmost and the second topmost row of σ are

$b = (b_1, \ldots, b_s)$ and $c = (c_1, \ldots, c_n)$ respectively, then $s \le n$ and the map $\gamma : \mathcal{T}(\lambda) \to \mathcal{T}(\lambda)$ is defined recursively as follows.

If $s = n$, we reverse b and add it on top of $\gamma(\tau)$. Let the result filling be $\gamma(\sigma)$.

If $s < n$, then we add the sequence $(0, \ldots, 0, b_s, \ldots, b_1)$ with $n - s$ zeros to the top of $\gamma(\tau)$. Denote the result filling by α and we will move all these zeros to the right of $b^r = (b_s, \ldots, b_1)$ by implementing the following steps: we start with the filling α and the rightmost zero of α,

(1) if $b_s > c_s$, we apply ϕ_{n-s} on α. If 0 and b_s are swapped by ϕ_{n-s}, we stop; otherwise we continue to apply t_{n-s} on $\phi_{n-s}(\alpha)$, that is, switch 0 and b_s;
(2) if $b_s \le c_s$, we apply $\rho_{n-s} \circ \phi_{n-s}$ on α. If 0 and b_s are swapped by ϕ_{n-s}, we continue to perform t_{n-s} on $\rho_{n-s} \circ \phi_{n-s}(\alpha)$; otherwise we stop.

As a result, this process produces a new filling where the rightmost zero in the top row is moved to column $(n-s+1)$. Repeat the procedure on this new filling and its rightmost zero, until all zeros in the top row are transported to the end. Remove these zeros and denote the resulting filling by $\gamma(\sigma)$. For both cases ($s = n$ and $s < n$), the topmost rows of σ and $\gamma(\sigma)$ are reverse of each other.

We will prove that (4.3)–(4.5) are satisfied by $\gamma(\sigma)$. Suppose that $\ell(\mu) = p$, let m denote the number of non-descents between rows p and $p+1$ of σ, that is, $m = \sum_{i=1}^{s} \chi(b_i \le c_i)$. Let $(x_n, \ldots, x_1)$ be the number of non-descents in the first n columns of α. Then the number of non-descents in the first n columns of $\gamma(\tau)$ is $(x_n - 1, \ldots, x_{s+1} - 1, x_s - \chi(b_s \le c_s), \ldots, x_1 - \chi(b_1 \le c_1))$, and the number of non-descents in the first n columns of σ is $(x_1, \ldots, x_s, x_{s+1} - 1, \ldots, x_n - 1)$. By definition of the major index, one sees that

$$\mathsf{maj}(\alpha) - \mathsf{maj}(\sigma) = \sum_{i=s+1}^{n} (p - 1 - (x_i - 1)) = \sum_{i=s+1}^{n} (p - x_i). \tag{6.1}$$

On the other hand, when we add $(n-s)$ zeros to the top row of σ, the number of inversion triples is increased by $m(n-s)$ as $\mathcal{Q}(b_i, c_i, 0) = 1$ if and only if $b_i \le c_i$. Therefore (5.1) gives that

$$\mathsf{quinv}(\alpha|_p^{p+1}) = \mathsf{inv}(\sigma|_p^{p+1}) + m(n-s) + \sum_{i=1}^{s} \chi(b_i \le c_i)(2i - n - 1) + \sum_{i=s+1}^{n} (2i - n - 1). \tag{6.2}$$

Let Ω_s be the block of α consisting of $0, b_s, c_{s+1}$ and c_s. We shall analyse the change of statistics quinv, $\mathcal{N}\mathsf{des}$ and maj by a close inspection on steps (1)–(2).

Case (I): For $b_s > c_s$, if ϕ_{n-s} swaps 0 and b_s of α, then (4.7) and (4.8) say that

$$\begin{aligned}
\mathsf{quinv}(\phi_{n-s}(\alpha)) &= \mathsf{quinv}(\alpha) + x_s - x_{s+1}, \\
\mathcal{N}\mathsf{des}(\phi_{n-s}(\alpha)) &= (\ldots, x_s, x_{s+1}, \ldots), \\
\mathsf{maj}(\phi_{n-s}(\alpha)) &= \mathsf{maj}(\alpha).
\end{aligned}$$

Case (II): Otherwise ϕ_{n-s} fixes entries 0 and b_s of α. In this case, ϕ_{n-s} also fixes entries c_{s+1}, c_s in the second topmost row and $\mathcal{Q}(0, c_{s+1}, c_s) \ne \mathcal{Q}(b_s, c_{s+1}, c_s)$ because of the followings. The block $\Omega_s \in \mathcal{B} \cup \mathcal{C}$ by Definition 3. If $\Omega_s \in \mathcal{B}$, then $0, b_s$ must be switched by ϕ_{n-s} because the first two rows of α contains exactly one descent block and $\varepsilon(\Omega_s)$ is applied according to the rules of ϕ_{n-s}. This excludes the possibility that $\Omega_s \in \mathcal{B}$ and we are left with $\Omega_s \in \mathcal{C}$. For $\Omega_s \in \mathcal{C}$, $\mathcal{Q}(0, c_{s+1}, c_s) \neq \mathcal{Q}(b_s, c_{s+1}, c_s)$ and Ω_s belongs to one component, thus ϕ_{n-s} must also fix entries c_{s+1}, c_s if it fixes entries $0, b_s$.

The condition $\mathcal{Q}(0, c_{s+1}, c_s) \neq \mathcal{Q}(b_s, c_{s+1}, c_s)$ implies that $c_s < b_s \leq c_{s+1}$. In step (1) we continue to implement t_{n-s} on $\phi_{n-s}(\alpha)$, which with the help of Lemma 9 leads to

$$\begin{aligned}
\mathsf{quinv}(t_{n-s} \circ \phi_{n-s}(\alpha)) &= \mathsf{quinv}(\alpha) + s - 1 + x_s - x_{s+1}, \\
\mathcal{N}\mathsf{des}(t_{n-s} \circ \phi_{n-s}(\alpha)) &= (\ldots, x_s, x_{s+1} + 1, \ldots), \\
\mathsf{maj}(t_{n-s} \circ \phi_{n-s}(\alpha)) &= \mathsf{maj}(\alpha) - 1.
\end{aligned}$$

Case (III): For $b_s \leq c_s$, we consider the topmost descent block between columns $n-s$ and $n-s+1$. Let β denote such block and $\varepsilon(\beta) = \rho^{\kappa}_{n-s}$. If ρ^{κ}_{n-s} is not a factor of ϕ_{n-s} or ρ^{κ}_{n-s} is a factor of ϕ_{n-s} but $\kappa < p+1$, then 0 and b_s are not interchanged by ϕ_{n-s} and we continue to perform ρ_{n-s} on $\phi_{n-s}(\alpha)$ that swaps 0 and b_s without crossing row κ. Besides, all blocks above row κ are neutral, so the number of non-descents is preserved by ρ_{n-s}. From Lemma 6, (4.7) and (4.8) we obtain

$$\begin{aligned}
\mathsf{quinv}(\rho_{n-s} \circ \phi_{n-s}(\alpha)) &= \mathsf{quinv}(\alpha) - 1 + x_s - x_{s+1}, \\
\mathcal{N}\mathsf{des}(\rho_{n-s} \circ \phi_{n-s}(\alpha)) &= (\ldots, x_s, x_{s+1}, \ldots), \\
\mathsf{maj}(\rho_{n-s} \circ \phi_{n-s}(\alpha)) &= \mathsf{maj}(\alpha).
\end{aligned}$$

Case (IV): For $b_s \leq c_s$ and ϕ_{n-s} has the factor ρ_{n-s}. Therefore the topmost two rows must be swapped by ϕ_{n-s}, which yields that $\mathcal{Q}(0, c_{s+1}, c_s) \neq \mathcal{Q}(b_s, c_{s+1}, c_s)$. It implies that $c_{s+1} < b_s \leq c_s$ and we continue to perform $t_{n-s} \circ \rho_{n-s}$ on $\phi_{n-s}(\alpha)$. Further, the number of non-descents in column $n-s$ is increased by one, while the one in column $n-s+1$ is decreased by one after applying ρ_{n-s} on $\phi_{n-s}(\alpha)$. In view of (5.10), (4.7), (4.8) and Lemma 9, we come to

$$\begin{aligned}
\mathsf{quinv}(t_{n-s} \circ \rho_{n-s} \circ \phi_{n-s}(\alpha)) &= \mathsf{quinv}(\alpha) - 1 - (s-1) + x_s - x_{s+1}, \\
\mathcal{N}\mathsf{des}(t_{n-s} \circ \rho_{n-s} \circ \phi_{n-s}(\alpha)) &= (\ldots, x_s, x_{s+1} - 1, \ldots), \\
\mathsf{maj}(t_{n-s} \circ \rho_{n-s} \circ \phi_{n-s}(\alpha)) &= \mathsf{maj}(\alpha) + 1.
\end{aligned}$$

Let α^i be the resulting filling after the rightmost zero of α is moved from column $n-s$ to column $n-s+i$ for $1 \leq i \leq s$, we will describe the changes of $\mathsf{quinv}, \mathcal{N}\mathsf{des}, \mathsf{maj}$ from α to α^1 for Cases (I)–(IV) in a unified manner.

The condition for Case (II) to occur is equivalent to $c_s < b_s \leq c_{s+1}$ and $\phi_{n-s}(\Omega_s) = \Omega_s$, and the one for Case (IV) is tantamount to $c_{s+1} < b_s \leq c_s$ and $\phi_{n-s}(\Omega_s) = \Omega^r_s$. Therefore, let $y_s = \chi(c_s < b_s \leq c_{s+1} \text{ and } \phi_{n-s}(\Omega_s) = \Omega_s) - \chi(c_{s+1} < b_s \leq c_s \text{ and } \phi_{n-s}(\Omega_s) = \Omega^r_s)$, thus

$$\begin{aligned}
\mathsf{quinv}(\alpha^1) - \mathsf{quinv}(\alpha) &= x_s - x_{s+1} + (s-1)y_s - \chi(b_s \leq c_s), \\
\mathcal{N}\mathsf{des}(\alpha^1) &= (x_n, \ldots, x_{s+2}, x_s, x_{s+1} + y_s, \ldots), \\
\mathsf{maj}(\alpha^1) - \mathsf{maj}(\alpha) &= -y_s.
\end{aligned}$$

Continue this process to produce α^j for $2 \le j \le s$, giving that

$$\mathsf{quinv}(\alpha^j) - \mathsf{quinv}(\alpha) = \sum_{i=s-j+1}^{s} (x_i - x_{s+1} - \chi(b_i \le c_i)) + (s-j) \sum_{i=s-j+1}^{s} y_i, \tag{6.3}$$

Take $j = s$, so that the last sum vanishes and we are led to

$$\mathsf{quinv}(\alpha^s) - \mathsf{quinv}(\alpha) = \sum_{i=1}^{s}(x_i - \chi(b_i \le c_i)) - sx_{s+1} = \sum_{i=1}^{s} x_i - sx_{s+1} - m. \tag{6.4}$$

Furthermore, let $z_1 = \sum_{i=1}^{s} y_i$, we find

$$\begin{aligned}\mathcal{N}\mathsf{des}(\alpha^s) &= (x_n, \ldots, x_{s+2}, x_s, \ldots, x_1, x_{s+1} + z_1, \ldots),\\ \mathsf{maj}(\alpha^s) - \mathsf{maj}(\alpha) &= -z_1.\end{aligned}$$

Observe that the topmost row of α^s is $(0, \ldots, 0, b_s, \ldots, b_1, 0)$ and let the second topmost row of α^s be $(d_{n-1}, \ldots, d_{s+1}, d_s, \ldots, d_1, d_0)$. We proceed by moving the rightmost zero of α^s to the second rightmost column via steps (1)–(2). Let α^{2s} be the resulting filling, we will deduce a formula similar to (6.3) and (6.4). First we have

$$\sum_{i=1}^{s} \chi(b_i \le d_i) = m + z_1,$$

which results from the fact that the number of non-descents between the topmost two rows of α is increased (or decreased) by one if and only if we are in Case (II) (or Case (IV)), that is when $y_i = 1$ (or $y_i = -1$). Define Γ_s to be the block of α^s consisting of $0, b_s, d_{s+1}, d_s$, let $y'_s = \chi(d_s < b_s \le d_{s+1}$ and $\phi_{n-s}(\Gamma_s) = \Gamma_s) - \chi(d_{s+1} < b_s \le d_s$ and $\phi_{n-s}(\Gamma_s) = \Gamma_s^r)$ and $z_2 = \sum_{i=1}^{s} y'_i$. Consequently,

$$\begin{aligned}\mathsf{quinv}(\alpha^{2s}) - \mathsf{quinv}(\alpha^s) &= \sum_{i=1}^{s}(x_i - \chi(b_i \le d_i)) - sx_{s+2} + z_2\\ &= \sum_{i=1}^{s} x_i - sx_{s+2} - m - z_1 + z_2.\end{aligned}$$

In addition, we have

$$\begin{aligned}\mathcal{N}\mathsf{des}(\alpha^{2s}) &= (x_n, \ldots, x_{s+3}, x_s, \ldots, x_1, x_{s+2} + z_2, x_{s+1} + z_1, \ldots),\\ \mathsf{maj}(\alpha^{2s}) - \mathsf{maj}(\alpha) &= -z_1 - z_2.\end{aligned}$$

Generally, for $1 \le i \le n-s$, let α^{is} be the resulting filling after the ith rightmost zero of $\alpha^{(i-1)s}$ is moved to the ith rightmost column and set $\alpha^0 = \alpha$. Let $(u_{s+1}, u_s, \ldots, u_1)$ be the sequence of entries right below $(0, b_s, \ldots, b_1)$ in $\alpha^{(i-1)s}$. Define Θ_s to be the block consisting of $0, b_s, u_s, u_{s+1}$, let $z_i = \sum_{j=1}^{s} \chi(u_j < b_j \le u_{j+1}$ and $\phi_{n-s}(\Theta_s) = \Theta_s) - \chi(u_{j+1} < b_j \le u_j$ and $\phi_{n-s}(\Theta_s) = \Theta_s^r)$. Then,

$$\sum_{j=1}^{s} \chi(b_j \le u_j) = m + \sum_{j=1}^{i-1} z_j.$$

$$\mathsf{quinv}(\alpha^{is}) - \mathsf{quinv}(\alpha^{(i-1)s}) = \sum_{j=1}^{s} x_j - sx_{s+i} - m - \sum_{j=1}^{i-1} z_j + z_i(i-1). \tag{6.5}$$

Further, the changes of $\mathcal{N}\mathsf{des}$ and maj are given by

$$\mathcal{N}\mathsf{des}(\alpha^{is}) = (x_n, \ldots, x_{s+i+1}, x_s, \ldots, x_1, x_{s+i} + z_i, \ldots, x_{s+1} + z_1, \ldots),$$
$$\mathsf{maj}(\alpha^{is}) - \mathsf{maj}(\alpha) = -\sum_{j=1}^{i} z_j. \tag{6.6}$$

Take $i = n - s$, and $\alpha^{(n-s)s}$ is the filling after all zeros are shifted to the end of the top row. Since $\gamma(\sigma)$ is obtained from $\alpha^{(n-s)s}$ by removing all zeros, we get $\mathsf{quinv}(\gamma(\sigma)) = \mathsf{quinv}(\alpha^{(n-s)s})$ and it follows from (6.5) that

$$\begin{aligned}\mathsf{quinv}(\gamma(\sigma)) - \mathsf{quinv}(\alpha) &= \mathsf{quinv}(\alpha^{(n-s)s}) - \mathsf{quinv}(\alpha)\\ &= \sum_{i=1}^{n-s}\left(\sum_{j=1}^{s} x_j - s x_{s+i} - m - \sum_{j=1}^{i-1} z_j + z_i(i-1)\right)\\ &= (n-s)\sum_{i=1}^{s} x_i - s\sum_{i=s+1}^{n} x_i - (n-s)m - \sum_{i=1}^{n-s} z_i(n-s-2i+1).\end{aligned} \tag{6.7}$$

In contrast, removing zeros of $\alpha^{(n-s)s}$ reduces the major index and the number of non-descents in the rightmost $n-s$ columns. To be precise,

$$\mathcal{N}\mathsf{des}(\gamma(\sigma)) = (x_s, \ldots, x_1, x_n + z_{n-s} - 1, \ldots, x_{s+1} + z_1 - 1, \ldots)$$
$$\mathsf{maj}(\gamma(\sigma)) - \mathsf{maj}(\alpha^{(n-s)s}) = -\sum_{i=1}^{n-s}(p - 1 - (x_{s+i} - 1 + z_i)). \tag{6.8}$$

Specializing $i = n - s$ in (6.6) and together with (6.8), we are led to

$$\mathsf{maj}(\gamma(\sigma)) = \mathsf{maj}(\alpha) - \sum_{i=1}^{n-s}(p - x_{s+i}) = \mathsf{maj}(\sigma),$$

where the last equality comes from (6.1). What remains is to discuss the change of quinv. Let us recall (5.2) that yields

$$\mathsf{inv}(\sigma) = \mathsf{inv}(\sigma|_p^{p+1}) + \mathsf{inv}(\tau) - \mathsf{inv}(c),$$
$$\mathsf{quinv}(\alpha) = \mathsf{quinv}(\alpha|_p^{p+1}) + \mathsf{quinv}(\gamma(\tau)) - \mathsf{inv}(c).$$

Plugging these two into (6.2) and (6.7) produces

$$\begin{aligned}\mathsf{quinv}(\gamma(\sigma)) = \mathsf{inv}(\sigma|_p^{p+1}) &+ \sum_{i=1}^{s}\chi(b_i \le c_i)(2i-n-1) + \sum_{i=s+1}^{n}(2i-n-1)\\ &+ \mathsf{inv}(\tau) + \kappa(\gamma(\tau)) - \mathsf{inv}(c) + (n-s)\sum_{i=1}^{s} x_i - s\sum_{i=s+1}^{n} x_i\\ &- \sum_{i=1}^{n-s} z_i(n-s-2i+1)\\ = \mathsf{inv}(\sigma) + \kappa(\gamma(\sigma)).\end{aligned}$$

The last equality holds because of the following expressions by Lemma 4.

$$\begin{aligned}
\kappa(\gamma(\tau)) &= \sum_{i=1}^{s}(x_i - \chi(b_i \le c_i))(2i-n-1) + \sum_{i=s+1}^{n}(x_i - 1)(2i-n-1) \\
&\quad + \sum_{i=2}^{p}(\mathsf{quinv}(\gamma(\tau)_i) - \mathsf{inv}(\gamma(\tau)_i^r), \\
\kappa(\gamma(\sigma)) &= \sum_{i=1}^{s} x_i(2i-s-1) + \sum_{i=1}^{n-s}(x_{i+s} + z_i - 1)(2i-(n-s)-1) \\
&\quad + \sum_{i=2}^{p}(\mathsf{quinv}(\gamma(\tau)_i) - \mathsf{inv}(\gamma(\tau)_i^r) \\
&= \sum_{i=1}^{s} x_i(2i-s-1) + \sum_{i=1}^{n-s}(x_{i+s} + z_i)(2i-n+s-1) \\
&\quad + \sum_{i=2}^{p}(\mathsf{quinv}(\gamma(\tau)_i) - \mathsf{inv}(\gamma(\tau)_i^r).
\end{aligned}$$

For the case $s = n$, it is easy to see that $\alpha = \gamma(\sigma)$, $\mathsf{maj}(\alpha) = \mathsf{maj}(\sigma)$, $\mathcal{N}\mathsf{des}(\sigma_1) = \mathcal{N}\mathsf{des}((\alpha_1)^r)$, and $\mathsf{quinv}(\alpha) = \mathsf{inv}(\sigma) + \kappa(\alpha)$ follows from the above argument. This completes the inductive proof. □

Example 3. *Let $\lambda = (7, 7, 5, 5, 5, 2)$ and σ is defined as below*

5	4					
9	3	6	1	3		
2	5	9	4	8		
3	9	7	3	5		
5	8	4	6	4	8	7
9	3	6	5	2	10	1

We will construct $\gamma(\sigma)$ and $\varphi(\sigma)$ step by step according to Theorem 2 and Theorem 8. Starting from the bottommost two rows, reverse them, add the sequence $(0, 0, 5, 3, 7, 9, 3)$ to the top, and move the two zeros to the end as follows, where each time the two columns under consideration *are coloured in green.*

0	**0**	**5**	3	7	9	3
7	**8**	**4**	6	4	8	5
1	**10**	**2**	5	6	3	9

→

0	5	**0**	**3**	7	9	3
7	4	**8**	**6**	4	8	5
1	2	**10**	**5**	6	3	9

→

0	5	3	**0**	**7**	9	3
7	4	8	**6**	**4**	8	5
1	2	5	**10**	**6**	3	9

→

0	5	3	7	**0**	**9**	3
7	4	8	6	**4**	**8**	5
1	2	5	10	**6**	**3**	9

→

0	5	3	7	9	**0**	**3**
7	4	8	6	8	**4**	**5**
1	2	5	10	3	**6**	**9**

→

0	**5**	3	7	9	3	0
7	**4**	8	6	8	4	5
1	**2**	5	10	3	6	9

→

5	**0**	**3**	7	9	3	0
4	**7**	**8**	6	8	4	5
1	**2**	**5**	10	3	6	9

→

5	3	**0**	**7**	9	3	0
4	7	**8**	**6**	8	4	5
1	2	**5**	**10**	3	6	9

→

5	3	7	**0**	**9**	3	0
4	7	6	**8**	**8**	4	5
1	2	10	**5**	**3**	6	9

→

5	3	7	9	**0**	**3**	0
4	7	6	8	**8**	**4**	5
1	2	10	5	**3**	**6**	9

→

5	3	7	9	3	0	0
4	7	6	8	4	8	5
1	2	10	5	6	3	9

Remove the two zeros, next add sequences $(8,4,9,5,2)$*,* $(3,1,6,3,9)$*, and* $(0,0,0,4,5)$ *successively to the top, obtaining the filling below.*

0	0	0	4	5		
3	1	6	3	9		
8	4	9	5	2		
5	3	7	9	3		
4	7	6	8	4	8	5
1	2	10	5	6	3	9

Transport the rightmost zero from column 3 *to column* 5*.*

0	**4**	5
6	**3**	9
9	**5**	2
7	**9**	3
6	**8**	4
10	**5**	6

→

4	**0**	**5**
6	**3**	**9**
9	**5**	**2**
7	**9**	**3**
6	**8**	**4**
5	**10**	**6**

→

4	5	0
6	9	3
9	2	5
7	3	9
6	8	4
5	10	6

Then move the zero from column 2 to column 4.

0	4	5	→	4	0	5	→	4	5	0
1	6	9		1	6	9		1	6	9
4	9	2		4	9	2		4	9	2
3	7	3		3	7	3		3	3	7
7	6	8		7	6	8		7	8	6
2	5	10		2	5	10		2	10	5

Finally shift the zero from column 1 to column 3.

0	4	5	→	4	0	5	→	4	5	0
3	1	6		3	1	6		3	6	1
8	4	9		4	8	9		4	8	9
5	3	3		3	5	3		3	3	5
4	7	8		7	4	8		7	8	4
1	2	10		1	2	10		1	10	2

Remove these zeros and $\gamma(\sigma)$ is generated.

4	5					
3	6	1	9	3		
4	8	9	2	5		
3	3	5	7	9		
7	8	4	6	4	8	5
1	10	2	5	6	3	9

We continue to apply θ on each rectangle of $\gamma(\sigma)$ independently.

4	5	→	4	5
3	6		6	3
4	8		8	4
3	3		3	3
7	8		7	8
1	10		10	1

1	9	3	→	1	3	9	→	1	3	9	→	1	3	9
9	2	5		9	5	2		5	9	2		5	9	2
5	7	9		5	7	9		5	7	9		5	7	9
4	6	4		4	4	6		4	4	6		4	4	6
2	5	6		2	6	5		6	2	5		6	2	5

8	5	→	8	5
3	9		9	3

Putting together these three rectangles gives $\varphi(\sigma)$ *as below.*

4	5					
6	3	1	3	9		
8	4	5	9	2		
3	3	5	7	9		
7	8	4	4	6	8	5
10	1	6	2	5	9	3

It is easy to check that $(\mathsf{inv}, \mathsf{maj})(\sigma) = (\mathsf{quinv}, \mathsf{maj})(\varphi(\sigma)) = (40, 33)$ *and note that* $\mathsf{quinv}(\sigma) \neq \mathsf{inv}(\varphi(\sigma))$ *as* $\mathsf{quinv}(\sigma) = 32$ *and* $\mathsf{inv}(\varphi(\sigma)) = 34$.

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References

[1] A. Ayyer, O. Mandelshtam, J.B. Martin, *Modified Macdonald polynomials and the multispecies zero-range process: I.* Algebraic Comb. 6, 243–284 (2023).

[2] A. Bhattacharya, T.V. Ratheesh, S. Viswanath, *Monomial expansions for q-Whittaker and modified Hall-Littlewood polynomials*, arXiv:2311.07904v2, 2023.

[3] T.V. Ratheesh, *Bijections between different combinatorial models for q–Whittaker and modified Hall–Littlewood polynomials*, arXiv: 2401.07481v2, 2024.

[4] S. Corteel, J. Haglund, O. Mandelshtam, S. Mason and L. Williams, personal communications, 2019.

[5] S. Corteel, J. Haglund, O. Mandelshtam, S. Mason and L. Williams, *Compact formulas for Macdonald polynomials and quasisymmetric Macdonald polynomials.* Sel. Math. New Ser. 28, 32 (2022).

[6] A. M. Garsia and M. Haiman, *A graded representation model for Macdonalds polynomials*, Proc. Nat. Acad. Sci. U.S.A. 90:8 (1993), 36073610.

[7] A. M. Garsia and J. Remmel, *Plethystic formulas and positivity for q, t-Kostka coefficients*, pp. 245262 in Mathematical essays in honor of Gian-Carlo Rota (Cambridge, MA, 1996), edited by B. E. Sagan and R. P. Stanley, Progress in Mathematics 161, Birkhäuser, Boston, 1998.

[8] A. M. Garsia and G. Tesler, *Plethystic formulas for Macdonald q, t-Kostka coefficients*, Adv. Math. 123:2 (1996), 144222.

[9] M. Haiman, *Hilbert schemes, polygraphs, and the Macdonald positivity conjecture*, J. Amer. Math. Soc. 14, 941–1006 (2001).

[10] J. Haglund, M. Haiman, N. Loehr, *A combinatorial formula for Macdonald polynomials.* J. Amer. Math. Soc. 18(3), 735–761 (2005).

[11] A. N. Kirillov and M. Noumi, *Affine Hecke algebras and raising operators for Macdonald polynomials*, Duke Math. J. 93:1 (1998), 139.

[12] F. Knop, *Integrality of two variable Kostka functions*, J. Reine Angew. Math. 482 (1997), 177189.

[13] N. Loehr, *Bijective Combinatorics*, Taylor and Francis/CRC Press (2011).

[14] N.A. Loehr, E. Niese, *A bijective proof of a factorization formula for specialized Macdonald polynomials.*, Ann. Comb. 16, 815–828 (2012).

[15] O. Mandelshtam, *A compact formula for the symmetric Macdonald polynomials*, arXiv: 2401.17223v2.

[16] S. Sahi, *Interpolation, integrality, and a generalization of Macdonalds polynomials*, Internat. Math. Res. Notices 10 (1996), 457471.

[17] R. Stanley, Enumerative combinatorics, volume 2, *Cambridge University Press*, (1999).

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