OUTSIDE NESTED DECOMPOSITIONS OF SKEW DIAGRAMS AND SCHUR FUNCTION DETERMINANTS

EMMA YU JIN

ABSTRACT. We describe the *thickened strips* and introduce the *outside nested decompositions* of any skew shape λ/μ. For any such decomposition $\Phi = (\Theta_1, \Theta_2, \ldots, \Theta_g)$ of the skew shape λ/μ where Θ_i is a thickened strip for every i, let r be the number of boxes that are contained in any two distinct thickened strips of Φ. Then we establish a determinantal formula of the function $p_{1^r}(X)s_{\lambda/\mu}(X)$ with the Schur functions of thickened strips as entries, where $s_{\lambda/\mu}(X)$ is the Schur function of the skew shape λ/μ and $p_{1^r}(X)$ is the power sum symmetric function indexed by the partition (1^r). This generalizes Hamel and Goulden's theorem on the outside decompositions of the skew shape λ/μ and our extension is motivated by the enumeration of m-strip tableaux, which was first counted by Baryshnikov and Romik via extending the transfer operator approach due to Elkies.

1. INTRODUCTION AND MAIN RESULTS

One of the most fundamental results on the symmetric functions is the determinantal expression of the Schur function $s_{\lambda/\mu}(X)$ for any skew shape λ/μ; see [11, 13]. The Jacobi-Trudi determinant [9, 6] and its dual [11, 6], the Giambelli determinant [5, 15] as well as the Lascoux and Pragacz's rim ribbon determinant [10, 16] are all of this kind. Hamel and Goulden [8] remarkably found that all above mentioned determinants for the Schur function $s_{\lambda/\mu}(X)$ can be unified through the concept of *outside decompositions* of the skew shape λ/μ.

In what follows all definitions will be postponed until subsection 1.3 and we first present Hamel and Goulden's theorem (Theorem 1).

Theorem 1 ([8]). *If the skew diagram of λ/μ is edgewise connected. Then, for any outside decomposition $\phi = (\theta_1, \theta_2, \ldots, \theta_g)$ of the skew shape λ/μ, it holds that*

$$(1.1) \qquad s_{\lambda/\mu}(X) = \det[s_{\theta_i \# \theta_j}(X)]_{i,j=1}^{g},$$

where $s_{\varnothing}(X) = 1$ and $s_{\theta_i \# \theta_j}(X) = 0$ if $\theta_i \# \theta_j$ is undefined.

Their proof is based on a lattice path construction and the Lindström-Gessel-Viennot methodology [6, 15]. In this paper we generalize the concept of outside decompositions even further, which is motivated by the enumeration of m-strip tableaux in [2].

When $m = 2k$, the enumeration of $2k$-strip tableaux is a direct consequence of the Lascoux and Pragacz's rim ribbon determinant [10, 16], or more broadly, Hamel and Goulden's

Corresponding author email: yu.jin@tuwien.ac.at.

theorem (Theorem 1). However, when $m = 2k+1$, any outside decomposition of $(2k+1)$-strip diagram with n columns consists of at least n strips (see subsection 3.2.2). So the order of the Jacobi-Trudi determinantal expression of $s_{\lambda/\mu}(X)$ can not be further reduced by applying Hamel and Goulden's theorem (Theorem 1). This motivates us to extend Hamel and Goulden's theorem.

1.1. **Our main results.** We introduce the concept of *outside nested decompositions* of the skew shape λ/μ and our first main result is a generalization of Theorem 1 with respect to any outside nested decomposition $\Phi = (\Theta_1, \Theta_2, \ldots, \Theta_g)$ of the skew shape λ/μ.

For any such decomposition $\Phi = (\Theta_1, \Theta_2, \ldots, \Theta_g)$ of the skew shape λ/μ where Θ_i is a thickened strip for every i, if r is the number of boxes that are contained in two distinct thickened strips of Φ. Then, our main theorem provides a determinantal formula of the function $p_{1^r}(X)s_{\lambda/\mu}(X)$ with the Schur functions of thickened strips as entries. The precise statement is the following.

Theorem 2. *If the skew diagram of λ/μ is edgewise connected. Then, for any outside nested decomposition $\Phi = (\Theta_1, \Theta_2, \ldots, \Theta_g)$ of the skew shape λ/μ, we set $r = \sum_{i=1}^{g} |\Theta_i| - |\lambda/\mu|$ that is the number of common special corners of Φ and we have*

$$(1.2) \qquad p_{1^r}(X)\, s_{\lambda/\mu}(X) = \det[s_{\Theta_i \# \Theta_j}(X)]_{i,j=1}^{g} \quad \text{where } p_{1^r}(X) = \Big(\sum_{i=1}^{\infty} x_i\Big)^r,$$

$s_{\varnothing}(X) = 1$ and $s_{\Theta_i \# \Theta_j}(X) = 0$ if $\Theta_i \# \Theta_j$ is undefined. The function $p_{1^r}(X)$ is the power sum symmetric function indexed by the partition (1^r) and $p_{1^r}(X) = 1$ if $r = 0$.

When $r = 0$ and all thickened strips Θ_i are strips, we retrieve Hamel and Goulden's theorem on the *outside decompositions* of the skew shape λ/μ. With the help of Theorem 2, it suffices to find an outside nested decomposition with minimal number of thickened strips in order to reduce the order of the determinantal expression of the Schur function $s_{\lambda/\mu}(X)$.

Let $|\lambda/\mu|$ and $f^{\lambda/\mu}$ denote the number of boxes contained in the skew shape λ/μ and the number of standard Young tableaux of shape λ/μ with the entries from 1 to $|\lambda/\mu|$ (similarly for $|\Theta_i \# \Theta_j|$ and $f^{\Theta_i \# \Theta_j}$). Then, by applying the exponential specialization on both sides of (1.2), one immediately gets

Corollary 3. *If the skew diagram of λ/μ is edgewise connected. Then, for any outside nested decomposition $\Phi = (\Theta_1, \Theta_2, \ldots, \Theta_g)$ of the skew shape λ/μ, we have*

$$(1.3) \qquad f^{\lambda/\mu} = |\lambda/\mu|!\, \det\big[(a_{i,j}!)^{-1} f^{\Theta_i \# \Theta_j}\big]_{i,j=1}^{g} \quad \text{where } a_{i,j} = |\Theta_i \# \Theta_j|,$$

$f^{\varnothing} = 1$ and $f^{\Theta_i \# \Theta_j} = 0$ if $\Theta_i \# \Theta_j$ is undefined.

Note that the parameter r vanishes in (1.3).

Our second main result is an enumeration of m-strip tableaux by applying Corollary 3, which provides another proof of Baryshnikov and Romik's results in [2]. Baryshnikov and Romik [2] counted m-strip tableaux via extending the transfer operator approach due to Elkies [4]. In their proof, it is of central importance that the transfer operator for m-strip tableaux can be diagonalized; see Theorem 6, 10, 11 in [2]. We believe that such

diagonalization of the transfer operator is in general closely related to the outside *nested* decompositions of the skew shape.

To prove Theorem 2, we use the bijection from semistandard Young tableaux to non-intersecting lattice paths in [8] and Stembridge's theorem [15] on non-intersecting lattice paths (which developed Lindström-Gessel-Viennot's approach). More precisely, our proof consists of three main steps:

semistandard Young tableaux $\rightarrow$ *separable double lattice paths* $\rightarrow$ *involution.*

In the first step we build a one-to-one correspondence between semistandard Young tableaux of thickened strip shape and *double lattice paths.* In the second step we introduce the *separable* sequences of double lattice paths and show that the generating function of all weighted separable sequences of double lattice paths is $p_{1^r}(X)s_{\lambda/\mu}(X)$. In the last step we construct a sign-reversing and weight-preserving involution f on all non-separable sequences of double lattice paths, so that only the separable ones contribute to the determinant $\det[s_{\Theta_i \# \Theta_j}(X)]$ in Theorem 2.

It turns out that thickened strips are the *maximal* small skew shapes such that the lattice path approach could work perfectly to establish a determinantal formula of Schur function. Furthermore, the proof from strips and outside decompositions respectively to thickened strips and outside nested decompositions works equally well for symplectic and orthogonal Schur functions, which are defined combinatorially in a manner similar to Schur functions. The determinantal formulas for symplectic and orthogonal Schur functions with respect to any outside decomposition have been discussed by Hamel in [7].

1.2. **Paper outline.** In subsection 1.3 and 1.4 we introduce all necessary notations and definitions. In Section 2 we prove Theorem 2 and Corollary 3. In Section 3 we introduce the notion of m-strip tableaux and count the number of m-strip tableaux.

1.3. **Partitions and symmetric functions.**

- A *partition* λ of n, denoted by $\lambda \vdash n$, is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$ and their sum is n. The non-zero λ_i are called the *parts* of λ and the number of parts is the length of λ, denoted by $\ell(\lambda)$.
- Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, the *standard diagram* of λ is a left-justified array of $\lambda_1 + \lambda_2 + \cdots + \lambda_m$ boxes with λ_1 in the first row, λ_2 in the second row, and so on.
- A *skew diagram* of λ/μ (also called *a skew shape* λ/μ) is the difference of two standard diagrams where $\mu \subseteq \lambda$. Note that the standard shape λ is just the skew shape λ/μ when $\mu = \varnothing$.
- The *content* of a box α in a skew shape λ/μ equals $t - s$ if the box α is in column t from the left and row s from the top of the skew shape λ/μ. We refer to box α as box (s, t) and (s, t) is called its *coordinate*. A *diagonal of content c* in a skew diagram is a set of boxes with content c in a skew diagram.
- A skew diagram 'starts' at a box (called *the starting box*) if that box is the bottommost and leftmost box in the skew diagram, and a skew diagram 'ends' at a box

(called *the ending box*) if that box is the topmost and rightmost box in the skew diagram.

- A *semistandard Young tableau* (resp. *standard Young tableau*) of skew shape λ/μ is a filling of the boxes of the skew diagram of λ/μ with positive integers such that the entries strictly increase down each column and weakly (resp. strictly) increase left to right across each row.

In a semistandard Young tableau T we use $T(\alpha)$ to represent the positive integer in the box α of T. The *Schur function*, $s_{\lambda/\mu}(X)$, in the variables $X = (x_1, x_2, \ldots)$, is given by

$$s_{\lambda/\mu}(X) = \sum_T \prod_{\alpha \in \lambda/\mu} x_{T(\alpha)},$$

where the summation is over all semistandard Young tableaux T of shape λ/μ and $\alpha \in \lambda/\mu$ means that α ranges over all boxes in the skew diagram of λ/μ. In particular, $s_\varnothing(X) = 1$. The *complete symmetric functions* $h_k(X)$ are defined by

$$h_k(X) = \sum_{1 \leq i_1 \leq \cdots \leq i_k} \prod_{j=i_1}^{i_k} x_j \ \text{ if } k \geq 1, \ h_0(X) = 1 \text{ and } h_k(X) = 0 \text{ if } k < 0.$$

The Jacobi-Trudi identity is a determinantal expression of Schur function $s_{\lambda/\mu}(X)$ in terms of complete symmetric functions $h_k(X)$; see [11, 13].

Theorem 4 (Jacobi-Trudi identity [9]). *Let λ/μ be a skew shape partition, let $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_k)$ have at most k parts. Then*

$$s_{\lambda/\mu}(X) = \det[h_{\lambda_i - \mu_j - i + j}(X)]_{i,j=1}^k.$$

The classical Aitken formula for the number of standard Young tableaux of skew shape can be directly obtained by applying the exponential specialization on the Jacobi-Trudi identity; see Chapter 7 of [13]. We denote by $|\lambda/\mu|$ the number of boxes contained in the skew diagram of λ/μ and denote by $f^{\lambda/\mu}$ the number of standard Young tableaux of shape λ/μ with the entries from 1 to $|\lambda/\mu|$.

Corollary 5 (Aitken formula). *Let λ/μ be a skew shape partition, let $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_k)$ have at most k parts. Then*

$$(1.4) \qquad f^{\lambda/\mu} = |\lambda/\mu|! \det \left[((\lambda_i - \mu_j - i + j)!)^{-1} \right]_{i,j=1}^k.$$

It is clear that the order of the determinant in the Jacobi-Trudi identity and in the Aitken formula equals the number $\ell(\lambda)$ of parts in λ. However, using (1.4) to compute $f^{\lambda/\mu}$ becomes difficult when the partitions λ and μ are large, even when their difference λ/μ is small.

1.4. **Outside nested decompositions.** We start with the strips and outside decompositions. Hamel and Goulden described the notion of an *outside decomposition* of the skew shape λ/μ, which generalizes Lascoux and Pragacz's rim ribbon decomposition [10]. With the help of Hamel and Goulden's theorem [8], for any skew shape λ/μ, one can reduce the

order of the determinant in the Jacobi-Trudi identity to the number of strips contained in any outside decomposition of skew shape λ/μ.

Two boxes are said to be edgewise connected if they share a common edge. A skew diagram θ is said to be *edgewise connected* if θ is an edgewise connected set of boxes.

Definition 1.1 (strip). A skew diagram θ is a *strip* if θ is edgewise connected and it contains no 2×2 blocks of boxes.

Remark 1.1. *The strips in Definition 1.1 are called 'border strips' by Macdonald* [11] *and are called 'ribbons' by Lascoux and Pragacz* [10]. *We adopt the name 'strips' from* [8].

Definition 1.2 (outside decomposition [8]). Suppose that $\theta_1, \theta_2, \ldots, \theta_g$ are strips of a skew diagram of λ/μ and every strip has a starting box on the left or bottom perimeter of the diagram and an ending box on the right or top perimeter of the diagram. Then we say the totally ordered set $\phi = (\theta_1, \theta_2, \ldots, \theta_g)$ is an *outside decomposition* of λ/μ if the union of these strips is the skew diagram of λ/μ and every two strips θ_i, θ_j in ϕ are disjoint, that is, θ_i and θ_j have no boxes in common.

Remark 1.2. *The rim ribbon decomposition of λ/μ introduced by Lascoux and Pragacz* [10] *is an outside decomposition with minimal number of strips; see* [14] *and* [17].

Example 1. *See Fig. 1.1 for an outside decomposition and two non-outside decompositions where all boxes are marked by black dots. The first two decompositions in Fig. 1.1 are not outside decompositions since the strip $\theta_1 = (5,1)$ of the left one has a starting box neither on the left nor on the bottom perimeter of the skew diagram and the strip $\theta_2 = (3)$ of the middle one has an ending box neither on the right nor on the top perimeter of the skew diagram.*
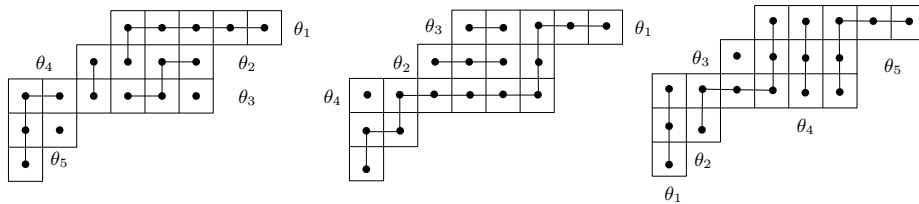

FIGURE 1.1. Two non-outside decompositions (left and middle) and one outside decomposition (right) of skew shape $(8,6,6,2,1)/(3,2)$.

We next introduce the notion of *thickened strips* and we will decompose the skew diagram of λ/μ into a sequence of thickened strips, in order to extend Hamel and Goulden's theorem [8] on the determinantal expression of the Schur function $s_{\lambda/\mu}(X)$.

Definition 1.3 (thickened strip). A skew diagram Θ is a thickened strip if Θ is edgewise connected and it neither contains a 3×2 block of boxes nor a 2×3 block of boxes.

Remark 1.3. *By definition the only difference between strips and thickened strips is that thickened strips could have some 2×2 blocks of boxes; see Fig. 1.2.*
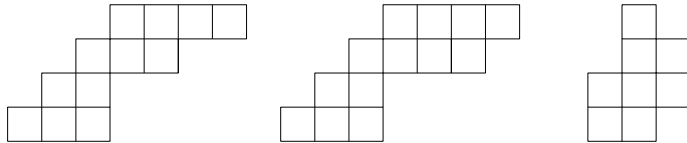
FIGURE 1.2. The left one is a thickened strip, while the middle one and the right one are not thickened strips.

We next define the *corners* and the *special corners* of a thickened strip Θ_i because in contrast to the outside decompositions, we allow two thickened strips in an outside nested decomposition to have special corners in common. In what follows, note that the box (s, t) always refers to the box with coordinate (s, t) in the skew diagram of λ/μ.

Definition 1.4. (corner, special corner) When a thickened strip Θ_i has more than one box, we define that *a corner* (s, t) of a thickened strip Θ_i is an upper corner or a lower corner, where *an upper corner* (s, t) of Θ_i is a box (s, t) such that neither the box $(s-1, t)$ nor the box $(s, t-1)$ is contained in Θ_i. Likewise, *a lower corner* (s, t) of Θ_i is a box (s, t) such that neither the box $(s+1, t)$ nor the box $(s, t+1)$ is contained in Θ_i. We say that a corner (s, t) of a thickened strip Θ_i is *special* if the corner (s, t) satisfies one of the following conditions:

(1) the corner (s, t) is the starting box or the ending box of Θ_i;
(2) the corner (s, t) is contained in a 2×2 block of boxes of Θ_i.

Example 2. *Consider the thickened strip in Fig. 1.2 (the left one), the only corner that is not special in this thickened strip is the box* $(2, 3)$.

Now we are ready to present the outside thickened strip decomposition.

Definition 1.5 (outside thickened strip decomposition). Suppose that $\Theta_1, \Theta_2, \ldots, \Theta_g$ are thickened strips in the skew diagram of λ/μ and every thickened strip has a starting box on the left or bottom perimeter of the diagram and an ending box on the right or top perimeter of the diagram. Then we say the totally ordered set $\Phi = (\Theta_1, \Theta_2, \ldots, \Theta_g)$ is an *outside thickened strip decomposition* of the skew diagram of λ/μ if the union of the thickened strips Θ_i of Φ is the skew diagram of λ/μ, and for all i, j, one of the following statements is true:

(1) two thickened strips Θ_i and Θ_j are disjoint, that is, Θ_i and Θ_j have no boxes in common;
(2) one thickened strips Θ_j is on the right side or the bottom side of the other thickened strip Θ_i and they have some special corners in common, where each common special corner (s, t) is a lower corner of Θ_i and an upper corner of Θ_j.

Every special corner of a thickened strip in Φ is called a *special corner of Φ* and every common special corner of any two distinct thickened strips of Φ is called a *common special corner of Φ*.

Remark 1.4. *If Θ_i has only one box (s,t) and box (s,t) is also a special corner of Θ_j. Then the outside thickened strip decomposition Φ is essentially the same as the one without Θ_i. So we exclude this scenario.*

Example 3. *Fig. 1.3 (middle, right) shows an outside thickened strip decomposition $\Phi = (\Theta_1, \Theta_2, \Theta_3)$ of the skew diagram of $(6,6,6,4)/(3,1)$ where the boxes $(4,1)$ and $(3,3)$ are the common special corners of Θ_2 and Θ_3. The box $(2,5)$ is the only common special corner of Θ_1 and Θ_2. In Fig. 1.3 every common special corner of Φ is marked by a black square, while other boxes are marked by black dots.*
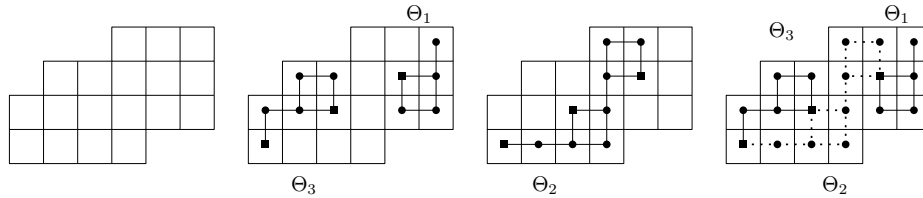
FIGURE 1.3. An outside thickened strip decomposition $(\Theta_1, \Theta_2, \Theta_3)$ (middle, right) of the skew diagram $(6,6,6,4)/(3,1)$ (left).

We observe that, unlike the strips in any outside decomposition, the thickened strips in any outside thickened strip decomposition Φ are *not necessarily nested*; see Definition 1.7. However, the *nested property* of thickened strips in an outside thickened strip decomposition is of central importance in the proof of Theorem 2. In view of this, we need to introduce the *enriched diagrams* and the *directions* of all boxes in the skew shape λ/μ to describe the nested property of thickened strips.

Definition 1.6 (enriched diagram). Suppose that $\Phi = (\Theta_1, \Theta_2, \ldots, \Theta_g)$ is an outside thickened strip decomposition of the skew shape λ/μ, for every i such that $1 \leq i \leq g$, and for box (s,t) that is the starting box or the ending box of Θ_i, we shall add new boxes to Θ_i according to the following rules:

    (1) if box (s,t) is a lower corner of Θ_i and an upper corner of some other thickened strip in Φ, we add boxes $(s, t-1), (s-1, t), (s-1, t-1)$ that are not contained in Θ_i to Θ_i;

    (2) if box (s,t) is an upper corner of Θ_i and a lower corner of some other thickened strip in Φ, we add boxes $(s, t+1), (s+1, t), (s+1, t+1)$ that are not contained in Θ_i to Θ_i,

where all the coordinates of new boxes are *relative* to the coordinates of the boxes in the skew diagram of λ/μ. We denote by $D(\Theta_i)$ the diagram after adding the new boxes to Θ_i and we call $D(\Theta_i)$ *an enriched thickened strip*. If neither the starting box nor the ending box of Θ_i satisfies 1 or 2, then $D(\Theta_i) = \Theta_i$. An *enriched diagram* $D(\Phi)$ is the union of all enriched thickened strips $D(\Theta_i)$ for every Θ_i of Φ.

Example 4. *In Fig. 1.3 the box $(4,1)$ contained in Θ_3 and Θ_2 is the only box that satisfies conditions (1) and (2) of Definition 1.6. So we add the boxes $(4,0), (3,0)$ to Θ_3 and add the boxes $(5,1), (5,2)$ to Θ_2; see Fig. 1.4 where all newly added boxes are colored grey.*
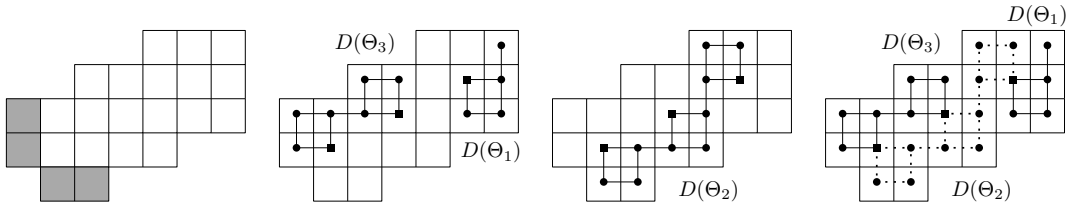
FIGURE 1.4. The enriched diagram $D(\Phi)$ (left) and the enriched thickened strips (middle, right) where $\Phi = (\Theta_1, \Theta_2, \Theta_3)$ is given in Fig. 1.3.

The enriched diagram $D(\Phi)$ may not be a skew diagram; see Fig. 1.4. With the help of enriched diagram $D(\Phi)$, one can define the directions of all boxes other than the special corners of Φ in the skew diagram of λ/μ. For every box (s, t) of the skew diagram of λ/μ, if box (s, t) is *not* a special corner of Φ, then box (s, t) is contained in only one thickened strip Θ_i of Φ. We may define the *direction* of box (s, t) in the enriched diagram $D(\Phi)$ according to the following rules:

(1) if both boxes $(s, t + 1)$ and $(s - 1, t)$ are contained in the enriched thickened strip $D(\Theta_i)$ of $D(\Phi)$, then we say the box (s, t) *goes right and up*;

(2) if not both boxes $(s - 1, t)$ and $(s, t + 1)$ are contained in $D(\Theta_i)$, then we say that the box (s, t) *goes right* or *goes up* if $(s, t + 1)$ or $(s - 1, t)$ is contained in $D(\Theta_i)$;

(3) if neither box $(s, t + 1)$ nor box $(s - 1, t)$ is contained in $D(\Theta_i)$, then box (s, t) must be the ending box of Θ_i, thus it must be on the top or right perimeter of the skew diagram of λ/μ, and we say that box (s, t) *goes up* if it is on the top perimeter of λ/μ and that box (s, t) *goes right* if it is on the right perimeter but not on the top perimeter of the skew diagram of λ/μ.

Definition 1.7 (outside nested decomposition)**.** An outside thickened strip decomposition Φ is an outside nested decomposition if $\Phi = (\Theta_1, \Theta_2, \ldots, \Theta_g)$ is *nested*, that is, for all c, one of the following statements is true:

(1) all boxes of content c all go right or all go up;

(2) all boxes of content c or all boxes of content $(c + 1)$ are all special corners of Φ.

Remark 1.5. *It should be noted that all boxes of content $(c + 1)$ are special corners of Φ if and only if all boxes of content c all go right and up. Definition 1.7 is analogous to the nested property of the strips in any outside decomposition where all boxes on the same diagonal of the skew shape λ/μ all go right or all go up; see* [3, 8]*.*

Example 5. *The outside thickened strip decomposition in Fig. 1.3 is an outside nested decomposition because all boxes on the diagonal of content $-3, 0, 3$ are all special corners, all boxes on the diagonal of content $1, 4, 5$ all go up, all boxes on the diagonal of content -2 all go right, and all boxes on the diagonal of content $-1, 2$ go right and up.*

Hamel and Goulden [8] defined a non-commutative operation $\#$ for every two strips of an outside decomposition $\phi = (\theta_1, \theta_2, \ldots, \theta_g)$ of the skew shape λ/μ, also when the skew shape λ/μ is edgewise disconnected. Subsequently, Chen, Yan and Yang [3] came up with the

notion of cutting strips so as to derive a transformation theorem for Hamel and Goulden's determinantal formula, in which one of the key ingredients is a bijection between the outside decompositions of a given skew diagram and the cutting strips.

Based on these previous work, we will extend the non-commutative operation $\#$ to every two thickened strips of an outside nested decomposition $\Phi = (\Theta_1, \Theta_2, \ldots, \Theta_g)$ of the skew shape λ/μ. In order to provide a simple definition of $\Theta_i \# \Theta_j$, we need to introduce the *thickened cutting strips*, which are called 'cutting strips' for any outside decomposition in [3].

Definition 1.8 (thickened cutting strips). The *thickened cutting strip* $H(\Phi)$ with respect to an outside nested decomposition $\Phi = (\Theta_1, \Theta_2, \ldots, \Theta_g)$ is a thickened strip obtained by successively superimposing the enriched thickened strips $D(\Theta_1), D(\Theta_2), \ldots, D(\Theta_g)$ of $D(\Phi)$ along the diagonals.

We say that *a box α of the thickened cutting strip $H(\Phi)$ has content c* if box α is on the diagonal of content c in the skew diagram of λ/μ and we represent each box of the thickened cutting strip $H(\Phi)$ as follows:

(1) box $[c]$ denotes the unique box of $H(\Phi)$ with content c;
(2) box $[c, +]$ and box $[c, -]$ denote the upper and the lower corner of $H(\Phi)$ with content c if they are contained in a 2×2 block of boxes in $H(\Phi)$.

Because of the nested property in Definition 1.7, the thickened cutting strip $H(\Phi)$ with respect to any outside nested decomposition Φ is a thickened strip.

Example 6. *Consider the outside nested decomposition $\Phi = (\Theta_1, \Theta_2, \Theta_3)$ in Fig. 1.3, the thickened cutting strip with respect to Φ is constructed in Fig. 1.5, where the dashed lines represent the diagonals of content $-3, 0, 3$ respectively.*
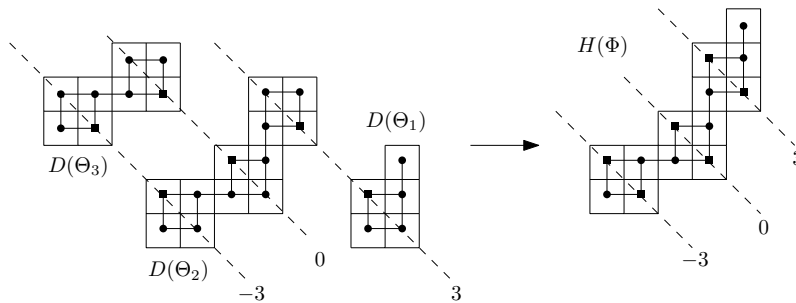

FIGURE 1.5. The thickened cutting strip $H(\Phi)$ (right) with respect to the outside nested decomposition $(\Theta_1, \Theta_2, \Theta_3)$ in Fig. 1.3 and the enriched thickened strips $D(\Theta_1), D(\Theta_2), D(\Theta_3)$ are given in Fig. 1.4.

Definition 1.9 ($\Theta_i \# \Theta_j$). If the skew diagram of λ/μ is edgewise connected, let $\Phi = (\Theta_1, \Theta_2, \ldots, \Theta_g)$ be an outside nested decomposition of skew shape λ/μ, and let $H(\Phi)$ be the thickened cutting strip with respect to Φ. For each thickened strip Θ_i in Φ, if c_i is the content of the starting box of Θ_i, the starting box $P(\Theta_i)$ of Θ_i is given as below:

(1) $p(\Theta_i) = [c_i]$ if the starting box is not a special corner of Φ;

(2) $p(\Theta_i) = [c_i, +]$ if the starting box is a special corner of Φ and an upper corner of Θ_i;

(3) $p(\Theta_i) = [c_i, -]$ if the starting box is a special corner of Φ and a lower corner of Θ_i,

where a special corner of Φ is defined at the end of Definition 1.5. Likewise, we denote the ending box of Θ_i by $q(\Theta_i)$ if we replace $p(\Theta_i)$ by $q(\Theta_i)$ and replace the starting box by the ending box from the above notations. Then Θ_i forms a segment of the thickened cutting strip $H(\Phi)$ starting with the box $p(\Theta_i)$ and ending with the box $q(\Theta_i)$, which is denote by $[p(\Theta_i), q(\Theta_i)]$. We may extend the notion to $[p(\Theta_j), q(\Theta_i)]$ in the following way:

(1) if $c_j < c_i$ or $p(\Theta_j) = q(\Theta_i)$, then $[p(\Theta_j), q(\Theta_i)]$ is a segment of $H(\Phi)$ starting with the box $p(\Theta_j)$ and ending with the box $q(\Theta_i)$;

(2) if $p(\Theta_j)$ and $q(\Theta_i)$ are in the same diagonal of $H(\Phi)$ and $p(\Theta_j) \neq q(\Theta_i)$, or $c_j = c_i + 1$, then $[p(\Theta_j), q(\Theta_i)] = \varnothing$;

(3) if $c_j > c_i + 1$, then $[p(\Theta_j), q(\Theta_i)]$ is undefined.

For any two thickened strips Θ_i and Θ_j of Φ, the thickened strip $\Theta_i \# \Theta_j$ is defined as $[p(\Theta_j), q(\Theta_i)]$.

Remark 1.6. *We only need to deal with the outside nested thickened strip decompositions of an edgewise connected skew diagram because the Schur function of any edgewise disconnected diagram is a product of Schur functions of edgewise connected components.*

Remark 1.7. *Since Φ is an outside nested decomposition, we can identify every thickened strip Θ_i as a segment of $H(\Phi)$ starting with the box $p(\Theta_i)$ and ending with the box $q(\Theta_i)$.*

Example 7. *Consider the outside nested decomposition $\Phi = (\Theta_1, \Theta_2, \Theta_3)$ in Fig. 1.3, one has $\Theta_1 \# \Theta_2 = [p(\Theta_2), q(\Theta_1)] = [[-3, +], [5]] = (5, 5, 5, 4, 4)/(4, 3, 3, 2)$, that is, a segment of the thickened cutting strip $H(\Phi)$ in Fig. 1.5 starting with box $[-3, +]$ and ending with box with content $[5]$. Similarly, the thickened strips obtained by the operation $\#$ are given below:*

$$\Theta_1 \# \Theta_3 = (4, 4, 4, 3, 3, 1)/(3, 2, 2, 1), \quad \Theta_2 \# \Theta_1 = (2, 2),$$
$$\Theta_2 \# \Theta_3 = (4, 4, 3, 3, 1)/(2, 2, 1), \qquad \Theta_3 \# \Theta_1 = \varnothing, \qquad \Theta_3 \# \Theta_2 = (4, 4)/(2).$$

2. Proof of Theorem 2 and Corollary 3

Since it is convenient to construct an involution in the context of lattice paths, we choose to represent semistandard Young tableaux of thickened strip shape in the language of lattice paths. Our proof of Theorem 2 consists of three main steps, which are respectively presented in subsections 2.1-2.3. We will prove Corollary 3 by using the exponential specializations of the Schur functions and power sum symmetric functions.

2.1. **From Semistandard Young tableaux to double lattice paths.** First we recall that $H(\Phi)$ is the thickened cutting strip which corresponds to Φ (see Definition 1.8) and $\Theta_i \# \Theta_j$ is given in Definition 1.9. For any i, j, for any given semistandard Young tableau $T_{\Theta_i \# \Theta_j}$ of shape $\Theta_i \# \Theta_j$, we introduce the corresponding *double lattice path* $P(u_j, v_i)$ and such construction $T_{\Theta_i \# \Theta_j} \mapsto P(u_j, v_i)$ is invertible.

Definition 2.1 (double lattice paths). Under the assumption of Theorem 2, for a given semistandard Young tableau $T_{\Theta_i \# \Theta_j}$ of shape $\Theta_i \# \Theta_j$, the starting point u_j and the ending point v_i of the *double lattice path* $P(u_j, v_i)$ are given as below:

(1) if the starting box (s, t) of Θ_j is a common special corner of Φ, and
   if box (s, t) is a lower corner of Θ_j, then $u_j = (t - s, 1)$;
   otherwise if box (s, t) is an upper corner of Θ_j, then $u_j = (t - s, \infty)$;
if the starting box (s, t) of Θ_j is not a common special corner of Φ, and
   if box (s, t) is on the left perimeter of the skew shape λ/μ, then $u_j = (t - s, 1)$;
   otherwise if box (s, t) is only on the bottom perimeter of the skew shape λ/μ, then $u_j = (t - s, \infty)$;
(2) if the ending box (μ, ν) of Θ_i is a common special corner of Φ, and
   if box (μ, ν) is a lower corner of Θ_i, then $v_i = (\nu - \mu + 1, 1)$;
   otherwise if box (μ, ν) is an upper corner of Θ_i, then $v_i = (\nu - \mu + 1, \infty)$;
if the ending box (μ, ν) of Θ_i is not a common special corner of Φ, and
   if box (μ, ν) is on the right perimeter of the skew shape λ/μ, then $v_i = (\nu - \mu + 1, \infty)$;
   otherwise if box (μ, ν) is only on the top perimeter of the skew shape λ/μ, then $v_i = (\nu - \mu + 1, 1)$.

The double lattice path $P(u_j, v_i)$ consists of four types of steps: an up-vertical step $\uparrow (0, 1)$, a down-vertical step $\downarrow (0, -1)$, a horizontal step $\rightarrow (1, 0)$ and a diagonal step $\searrow (1, -1)$, which satisfy the conditions

(3) a down-vertical step $(0, -1)$ must not precede an up-vertical step $(0, 1)$ and must not precede a horizontal step $(1, 0)$;
(4) an up-vertical step $(0, 1)$ must not precede a down-vertical step $(0, -1)$ and must not precede a diagonal step $(1, -1)$.

If box α of content c has entry q in $T_{\Theta_i \# \Theta_j}$, then we put a horizontal step from (c, q) to $(c + 1, q)$ if one of the following are true:

(5) a box of content $c - 1$ is to the left of box α in $\Theta_i \# \Theta_j$;
(6) box α is the starting box of Θ_j and $u_j = (c, 1)$.

If box α of content c has entry q in $T_{\Theta_i \# \Theta_j}$, then we put a diagonal step from $(c, q + 1)$ to $(c + 1, q)$ if one of the following are true:

(7) a box of content $c - 1$ is right below box α in $\Theta_i \# \Theta_j$;
(8) box α is the starting box of Θ_j and $u_j = (c, \infty)$.

Finally after we connect all non-vertical steps by up-vertical and down-vertical steps, we get the double lattice path $P(u_j, v_i)$.

Remark 2.1. *If $\Theta_i \# \Theta_j = \varnothing$, then according to 5-8 of Definition 2.1, $P(u_j, v_i)$ has no non-vertical steps. If $\Theta_i \# \Theta_j$ is undefined, then the starting point u_j is on the right hand side of the ending point v_i, so by Definition 2.1 there exist no double lattice paths from u_j to v_i, that is, $P(u_j, v_i)$ is also undefined.*

By construction all starting points and ending points are all *distinct*. Once the starting point u_j and the ending point v_i are chosen, the shape of any double lattice path $P(u_j, v_i)$ is fixed, that is, whether any non-vertical step of $P(u_j, v_i)$ is horizontal or diagonal, is determined by $\Theta_i \# \Theta_j$.

Furthermore, we note that the double lattice path $P(u_j, v_i)$ records the entries of $T_{\Theta_i \# \Theta_j}$ via the y-th coordinates of all ending points in non-vertical steps, namely, the point $(c+1, q)$ is the ending point of some non-vertical step of $P(u_j, v_i)$ if and only if a box of content c has entry q in $T_{\Theta_i \# \Theta_j}$. So the construction $T_{\Theta_i \# \Theta_j} \mapsto P(u_j, v_i)$ is a bijection, which allows us to identify the Schur function $s_{\Theta_i \# \Theta_j}(X)$ as the generating function of all weighted double lattice paths from u_j to v_i in subsection 2.3.

Example 8. *For $i = 1, 2, 3$, consider the thickened strips in Fig. 1.3, the double lattice path $P(u_2, v_3)$ of the thickened strip tableau $T_{\Theta_3 \# \Theta_2}$ is given in Fig. 2.1 where all integers represent the y-th coordinates of all ending points from the non-vertical steps in $P(u_2, v_3)$. We have discussed the shape of $\Theta_3 \# \Theta_2$ in Example 7. Since the starting box $p(\Theta_2) = [-3, +]$ of $\Theta_3 \# \Theta_2$ is an upper corner of Θ_2, according to condition (1) in Definition 2.1, the starting point u_2 is $(-3, \infty)$ and we put a diagonal step from $(-3, 3)$ to $(-2, 2)$ in Fig. 2.2 because of condition (8) in Definition 2.1. Similarly, since the ending box of $\Theta_3 \# \Theta_2$ is $q(\Theta_3) = [1]$, the ending point v_3 is $(2, 1)$.*

In addition, the corresponding double lattice path $P(u_1, v_3)$ of the empty thickened strip tableau $T_{\Theta_3 \# \Theta_1} = T_\varnothing$ consists of only vertical steps from $u_1 = (2, \infty)$ to $v_3 = (2, 1)$.
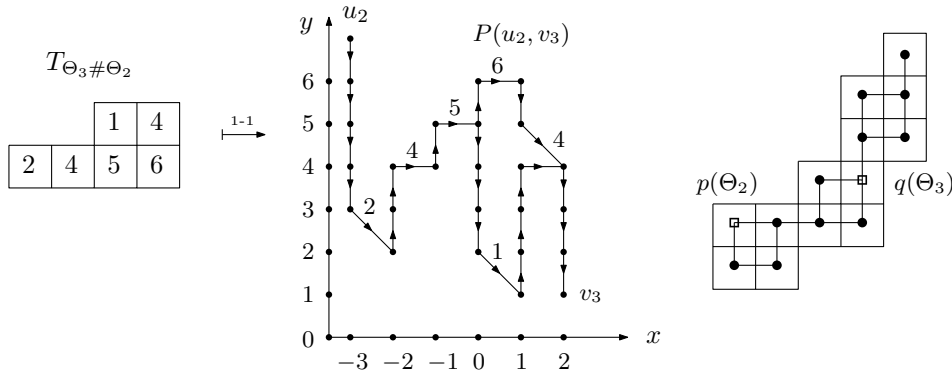

FIGURE 2.1. A thickened strip tableau $T_{\Theta_3 \# \Theta_2}$ (left), the corresponding double lattice path $P(u_2, v_3)$ (middle) and the thickened cutting strip $H(\Phi)$ where the starting box $p(\Theta_2) = [-3, +]$ and the ending box $q(\Theta_3) = [1]$ of $\Theta_3 \# \Theta_2$ are marked with empty squares (right).

With the bijection $T_{\Theta_i \# \Theta_j} \mapsto P(u_j, v_i)$ we can establish the bijection between semistandard Young tableaux of skew shape λ / μ and *non-crossing* g-tuples of double lattice paths in Proposition 7.

Definition 2.2 (non-crossing). For any $\pi = \pi_1 \pi_2 \cdots \pi_g \in S_g$, let

$$(2.1) \qquad (P(u_{\pi_1}, v_1), P(u_{\pi_2}, v_2), \ldots, P(u_{\pi_g}, v_g))$$

be a g-tuple of double lattice paths. Then (2.1) is *non-crossing* if for any i and j, $P(u_{\pi_i}, v_i)$ and $P(u_{\pi_j}, v_j)$ are non-crossing. This holds if one of 1, 2 is true.

(1) $P(u_{\pi_i}, v_i)$ and $P(u_{\pi_j}, v_j)$ are non-intersecting, that is, have no points in common;

(2) $P(u_{\pi_j}, v_j)$ is on the top side of $P(u_{\pi_i}, v_i)$ and they have some points in common, where each common point $(c+1, q)$ occurs only when one diagonal step of $P(u_{\pi_j}, v_j)$ and one horizontal step of $P(u_{\pi_i}, v_i)$ end at the same point $(c + 1, q)$.

Otherwise $P(u_{\pi_i}, v_i)$ and $P(u_{\pi_j}, v_j)$ are crossing and (2.1) is crossing. If (2.1) is non-crossing, we call every common point of any two double lattice paths in (2.1) a *touchpoint* of (2.1).

Example 9. *The triple $(P(u_1, v_1), P(u_2, v_2), P(u_3, v_3))$ of double lattice paths given in Fig. 2.2 where the y-coordinates of u_1, v_1, u_2 are all infinity, is non-crossing and all touchpoints have coordinates $(-2, 3), (1, 4), (4, 3)$.*
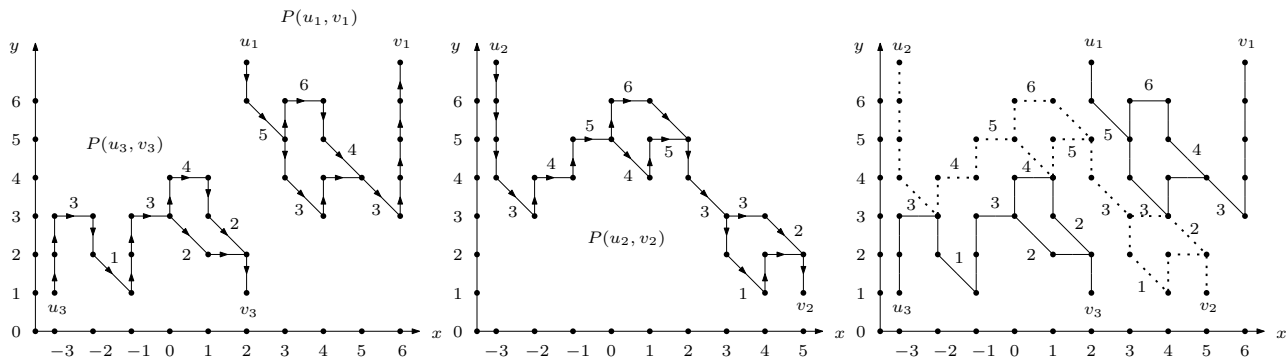
FIGURE 2.2. Three double lattice paths where each $P(u_i, v_i)$ uniquely corresponds to the semistandard Young tableau T_{Θ_i} in Fig. 2.3.

The lemma below actually verifies the condition of Stembridge's theorem on the non-intersecting lattice paths [15], which developed Lindström-Gessel-Viennot lattice paths approach [6]. Though Stembridge considered only the non-intersecting lattice paths, his theorem is still applicable to the non-crossing double lattice paths.

Lemma 6. *If a g-tuple (2.1) of double lattice path is non-crossing, then π must be the identity permutation, that is, $\pi = id = (1)(2) \cdots (g)$.*

The proof is the same as the one for the non-intersecting lattice paths in [8]. To make the paper self-contained, we put the proof of Lemma 6 in the Appendix.

Proposition 7. *Under the assumption of Theorem 2, there is a bijection between semistandard Young tableaux of skew shape λ/μ and non-crossing g-tuples of double lattice paths with r touchpoints.*

Proof. For a semistandard Young tableau T of the skew shape λ/μ, we can express T as a g-tuple $(T_{\Theta_1}, T_{\Theta_2}, \ldots, T_{\Theta_g})$ of thickened strip tableaux where T_{Θ_i} is T that is restricted to

the thickened strip shape Θ_i. By the bijection $T_{\Theta_i} \mapsto P(u_i, v_i)$ in Definition 2.1, one gets a g-tuple

$$(2.2) \qquad\qquad (P(u_1, v_1), P(u_2, v_2), \ldots, P(u_g, v_g))$$

of double lattice paths. The fact that (2.2) is non-crossing follows from the fact that all entries of boxes on the same diagonal of T are strictly increasing from the top-left side to the bottom-right side. The map $(T_{\Theta_1}, T_{\Theta_2}, \ldots, T_{\Theta_g}) \mapsto$ (2.2) is a bijection because, for any i and j, two double lattice paths $P(u_i, v_i)$ and $P(u_j, v_j)$ are non-intersecting if and only if two thickened strip tableaux T_{Θ_i} and T_{Θ_j} are disjoint. Furthermore, $P(u_j, v_j)$ is on the top side of $P(u_i, v_i)$ such that the diagonal step of $P(u_j, v_j)$ and the horizontal step of $P(u_i, v_i)$ end at the same point $(c+1, q)$ if and only if the box of content c and with entry q in T, is an upper corner of Θ_j and a lower corner of Θ_i. Since there are r common special corners of Φ, there are r touchpoints of (2.2). □

Example 10. *Consider the semistandard Young tableau $T = (T_{\Theta_1}, T_{\Theta_2}, T_{\Theta_3})$ of skew shape $(6, 6, 6, 4)/(3, 1)$ in Fig. 2.3, the corresponding triple of double lattice paths $P(u_i, v_i)$ under the bijection $T_{\Theta_i} \mapsto P(u_i, v_i)$ is displayed in Fig. 2.2 where the y-coordinates of u_1, v_1, u_2 are all infinity.*
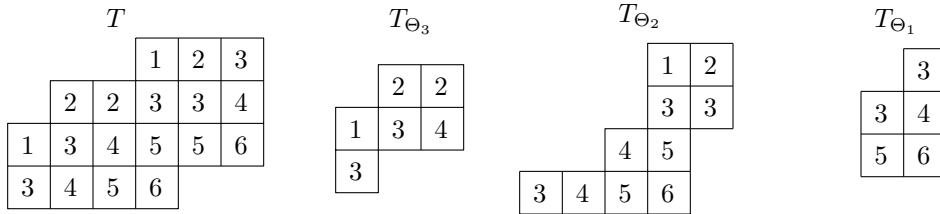
FIGURE 2.3. A semistandard Young tableau T which is equivalent to a triple $(T_{\Theta_1}, T_{\Theta_2}, T_{\Theta_3})$ of thickened strip tableaux and $\Phi = (\Theta_1, \Theta_2, \Theta_3)$ is given in Fig. 1.3.

2.2. **Count the separable sequences of double lattice paths.** We first introduce *separable* g-tuples P of double lattice paths in Definition 2.3 and then prove such g-tuples P of double lattice paths are in bijection with all pairs $(P, \{a_i\}_{i=1}^r)$ where P is a non-crossing g-tuple of double lattice paths and $\{a_i\}_{i=1}^r$ is a sequence of r positive integers in Proposition 8.

Definition 2.3 (separable double lattice paths)**.** For any $\pi \in S_g$, let

$$(2.3) \qquad\qquad \mathsf{P} = (\mathsf{P}(u_{\pi_1}, v_1), \mathsf{P}(u_{\pi_2}, v_2), \ldots, \mathsf{P}(u_{\pi_g}, v_g))$$

be a g-tuple of double lattice paths. For all $1 \le i \le g$, if $\mathsf{P}(u_{\pi_i}, v_i)$ has a point on line $x = c$, we define the unique *c-point* of $\mathsf{P}(u_{\pi_i}, v_i)$, which is the ending point of the non-vertical step of $\mathsf{P}(u_{\pi_i}, v_i)$ between lines $x = c - 1$ and $x = c$, or the starting point of $\mathsf{P}(u_{\pi_i}, v_i)$ on line $x = c$. Then P is *separable* if the following are true:

(1) for all c such that neither c nor $c-1$ is the content of some special corner of Φ, any two double lattice paths in P are *not intersecting* on line $x = c$;

(2) for all c such that c is the content of some special corner of Φ, for all pairs $(c, [i, j])$ such that the c-point of $\mathsf{P}(u_{\pi_i}, v_i)$ is *below* the one from $\mathsf{P}(u_{\pi_j}, v_j)$ and there is no other c-points in between,

- either $\mathsf{P}(u_{\pi_j}, v_j)$ after step (b) or $\mathsf{P}(u_{\pi_i}, v_i)$ after step (d) is a double lattice path, where steps $(b), (d)$ are given by

(b) shift the diagonal step between lines $x = c, x = c + 1$ to $(c, q + 1) \searrow (c + 1, q)$ if $(c + 1, q)$ is the ending point of a horizontal step from $P(u_{\pi_i}, v_i)$; change the vertical steps of $P(u_{\pi_j}, v_j)$ on lines $x = c$ and $x = c + 1$ so that they connect to the new diagonal step;

(d) shift the horizontal step between lines $x = c, x = c + 1$ to $(c, p) \to (c + 1, p)$ if $(c + 1, p)$ is the ending point of a diagonal step from $P(u_{\pi_j}, v_j)$; change the vertical steps of $P(u_{\pi_i}, v_i)$ on lines $x = c$ and $x = c + 1$ so that they connect to the new horizontal step.

Example 11. *For the outside nested decomposition $\Phi = (\Theta_1, \Theta_2, \Theta_3)$ in Fig. 1.3, the triple P of double lattice paths in Fig. 2.5 is separable. There are three pairs $(-3, [3, 2])$, $(0, [3, 2])$, $(3, [2, 1])$ satisfying condition (2) of Definition 2.3.*

For instance, the pair $(-3, [3, 2])$ satisfies condition (2) of Definition 2.3, because for $c = -3$, the c-points of $\mathsf{P}(u_3, v_3)$ and $\mathsf{P}(u_2, v_2)$ are respectively $u_3 = (-3, 1)$ and $u_2 = (-3, \infty)$. There is no other c-points in between. The double lattice path $\mathsf{P}(u_2, v_2)$, after shifting the diagonal step $(-3, 4) \searrow (-2, 3)$ to $(-3, 6) \searrow (-2, 5)$ and adjusting the vertical steps on lines $x = -3, -2$, is not a double lattice path; see the right one of Fig. 2.4, while $\mathsf{P}(u_3, v_3)$ after step (d) is a double lattice path, namely, shift the horizontal step $(-3, 5) \to (-2, 5)$ to $(-3, 3) \to (-2, 3)$ and adjust the vertical steps on lines $x = -3, -2$. See the left one of Fig. 2.4.
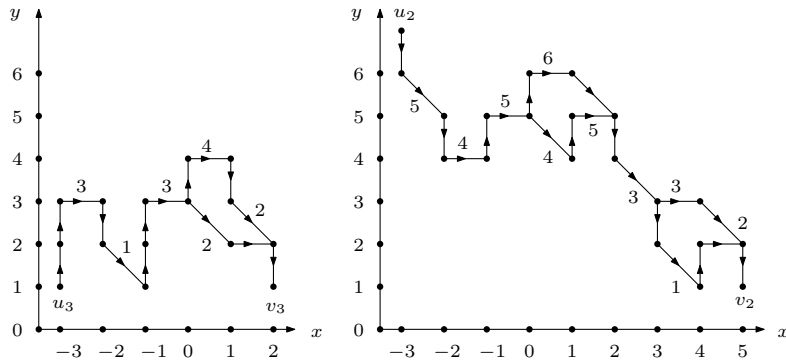

FIGURE 2.4. A double lattice path (left) and a non-double lattice path (right). The right one is not a double lattice path because a down-vertical step $(-2, 5) \downarrow (-2, 4)$ precedes a horizontal step $(-2, 4) \to (-1, 4)$, which fails to satisfy condition (3) of Definition 2.1.

Proposition 8. *Under the assumption of Theorem 2, for any fixed total order of all points in the 2-dimensional $\mathbb{N} \times \mathbb{N}$ grid, there is a bijection between all separable g-tuples P of double lattice paths with precisely r pairs $(c, [i, j])$ in condition (2) of Definition 2.3, and all pairs $(P, \{a_i\}_{i=1}^r)$ where $\{a_i\}_{i=1}^r$ is a sequence of r positive integers and P is a non-crossing g-tuple of double lattice paths with r distinct touchpoints.*

Furthermore, if P given in (2.3) is separable, then π must be the identity permutation, that is, $\pi = (1)(2) \cdots (g)$.

Proof. The map $\mathsf{P} \mapsto (P, \{a_i\}_{i=1}^r)$ is given as follows. For a given separable g-tuple (2.3) of double lattice paths with exactly r pairs $(c, [i, j])$ in condition (2) of Definition 2.3, we choose any total order of such pairs and perform the following procedure one by one: for $1 \leq s \leq r$, suppose that $(c, [i, j])$ is the s-th pair. Then,

(1) if $\mathsf{P}(u_{\pi_j}, v_j)$ after step (b) is a double lattice path, assume that the diagonal step of $\mathsf{P}(u_{\pi_j}, v_j)$ between lines $x = c, x = c+1$ ends at $(c+1, p)$. Then we set $a_s := p$ and replace $\mathsf{P}(u_{\pi_j}, v_j)$ by the new double lattice path after step (b);

(2) otherwise, assume that the horizontal step of $\mathsf{P}(u_{\pi_i}, v_i)$ between lines $x = c, x = c+1$ ends at $(c+1, q)$. Then we replace $\mathsf{P}(u_{\pi_i}, v_i)$ by the new double lattice path after step (d) and set $a_s := q$.

Note that after each replacement, we get a new g-tuple of double lattice paths, then we repeat procedure $(1), (2)$ on the next pair and so on. In each replacement, two double lattice paths become non-crossing between lines $x = c$ and $x = c+2$ where a touchpoint is located on line $x = c+1$, and this touchpoint is the s-th touchpoint in the total order, if and only if it is produced by the s-th pair $(c, [i, j])$ from the above procedure.

We denote by $P(u_{\pi_i}, v_i)$ the double lattice path after we finish the above procedure on $\mathsf{P}(u_{\pi_i}, v_i)$ for all i. Together with condition (1) of Definition 2.3, we find that the g-tuple of double lattice paths $P(u_{\pi_i}, v_i)$ must be non-crossing. In view of Lemma 6, $\pi = \mathrm{id} = (1)(2) \cdots (g)$. So $\mathsf{P} = (\mathsf{P}(u_1, v_1), \mathsf{P}(u_2, v_2), \ldots, P(u_g, v_g))$.

The reverse map $(P, \{a_i\}_{i=1}^r) \mapsto \mathsf{P}$ is given as follows. For a non-crossing g-tuple

$$P = (P(u_1, v_1), P(u_2, v_2), \ldots, P(u_g, v_g))$$

of double lattice paths with r touch points $\{(p_i, q_i)\}_{i=1}^r$, let $\{a_i\}_{i=1}^r$ be a sequence of positive integers. Then we perform the following procedure one by one: suppose that the diagonal step of $P(u_{s_i}, v_{s_i})$ and the horizontal step of $P(u_{t_i}, v_{t_i})$ intersect at the point (p_i, q_i). Then

(3) if $P(u_{s_i}, v_{s_i})$, after step (e), is still a double lattice path, then we replace $P(u_{s_i}, v_{s_i})$ by this new double lattice path.

(e) Shift the diagonal step $(p_i - 1, q_i + 1) \searrow (p_i, q_i)$ to $(p_i - 1, a_i + 1) \searrow (p_i, a_i)$ in $P(u_{s_i}, v_{s_i})$; change the vertical steps on lines $x = p_i - 1$ and $x = p_i$ so that they connect to the new diagonal step;

(4) otherwise $a_i > q_i$ and we replace $P(u_{t_i}, v_{t_i})$ by the new double lattice path after step (h).

(h) Shift the horizontal step $(p_i - 1, q_i) \to (p_i, q_i)$ to $(p_i - 1, a_i) \to (p_i, a_i)$ in $P(u_{t_i}, v_{t_i})$; change the vertical steps on lines $x = p_i - 1$ and $x = p_i$ so that they connect to the new horizontal step.

Note that after each replacement, we get a new g-tuple of double lattice paths, then we repeat procedure $(3), (4)$ on the next touchpoint and so on. Finally we retrieve the g-tuple P of double lattice paths after we finish the above procedure for all touchpoints $\{(p_i, q_i)\}_{i=1}^r$. It is easy to verify that P is separable by Definition 2.3. Consequently, $\mathsf{P} \mapsto (P, \{a_i\}_{i=1}^r)$ is a bijection. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Example 12. *Consider the pair $(P, \{a_i\}_{i=1}^3)$ where P is a non-crossing triple of double lattice paths in Fig. 2.2 and $\{a_i\}_{i=1}^3 = (5, 3, 5)$, we arrange the touchpoints in the sequence $\{(p_i, q_i)\}_{i=1}^3 = ((-2, 3), (1, 4), (4, 3))$, the corresponding separable triple P of double lattice paths, under the inverse map $(P, \{a_i\}_{i=1}^3) \mapsto \mathsf{P}$ in Proposition 8, is shown in Fig. 2.5.*
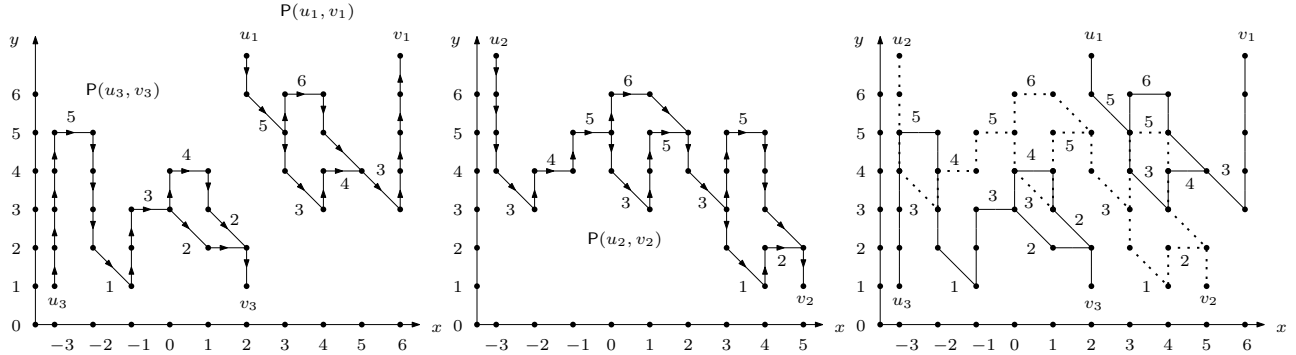
FIGURE 2.5. A separable triple of double lattice paths.

Example 13. *The triple of double lattice paths in Fig. 2.6 is* not *separable. For $c = 2$, $\mathcal{P}(u_1, v_3)$ intersects $\mathcal{P}(u_3, v_2)$ and $\mathcal{P}(u_2, v_1)$ on line $x = 2$. For $c = 3$, we consider the pair $(3, [2, 1])$ where neither $\mathcal{P}(u_2, v_1)$ after step (b) nor $\mathcal{P}(u_3, v_2)$ after step (d) is a double lattice path. This is also true for the pair $(0, [1, 2])$ when $c = 0$.*
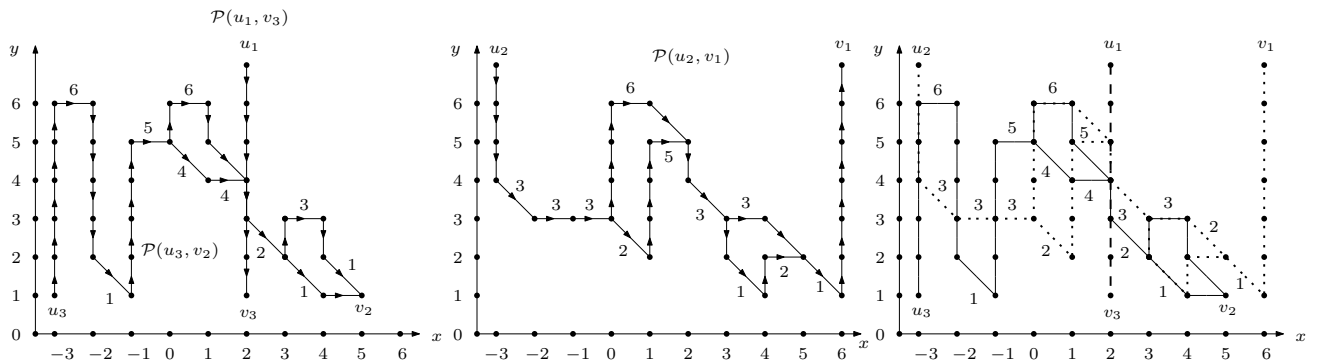
FIGURE 2.6. A non-separable triple of double lattice paths.

2.3. Construct the involution. For any permutation $\pi = \pi_1 \pi_2 \cdots \pi_g \in S_g$, the *inversion* of π is $\mathrm{inv}(\pi) = |\{(i,j) : \pi_i > \pi_j, i < j\}|$ and we may interpret the determinant in Theorem 2 as

$$(2.4) \qquad \det[s_{\Theta_i \# \Theta_j}(X)]_{i=1}^g = \sum_{\pi \in S_g} (-1)^{\mathrm{inv}(\pi)} \prod_{i=1}^g s_{\Theta_i \# \Theta_{\pi_i}}(X).$$

In view of the bijective construction $T_{\Theta_i \# \Theta_{\pi_i}} \mapsto P(u_{\pi_i}, v_i)$ in Definition 2.1, the generating function $s_{\Theta_i \# \Theta_{\pi_i}}(X)$ is

$$s_{\Theta_i \# \Theta_{\pi_i}}(X) = \sum_{P(u_{\pi_i}, v_i)} w(P(u_{\pi_i}, v_i)) := \sum_{P(u_{\pi_i}, v_i)} \prod_s v_s$$

where the sum ranges from all double lattice paths $P(u_{\pi_i}, v_i)$ from u_{π_i} to v_i and the weight $w(P(u_{\pi_i}, v_i))$ on the double lattice path $P(u_{\pi_i}, v_i)$ is the product of v_s, which ranges from all ending points s of non-vertical steps in $P(u_{\pi_i}, v_i)$ and v_s represents the weight on such point s. The *weight* v_s is defined as follows:

For each ending point $s = (a,b)$ of some non-vertical step, we assign it with weight $v_s = x_b$. For any other point, we assign it with weight 1.

Consequently (2.4) becomes

$$\det[s_{\Theta_i \# \Theta_j}(X)]_{i=1}^g = \sum_{\pi \in S_g} (-1)^{\mathrm{inv}(\pi)} \prod_{i=1}^g \sum_{P(u_{\pi_i}, v_i)} w(P(u_{\pi_i}, v_i))$$

$$(2.5) \qquad = \sum_{\pi \in S_g} \sum_P (-1)^{\mathrm{inv}(\pi)} \prod_{i=1}^g w(P(u_{\pi_i}, v_i))$$

where the last sum runs over all g-tuples $P = (P(u_{\pi_1}, v_1), P(u_{\pi_2}, v_2), \ldots, P(u_{\pi_g}, v_g))$ of double lattice paths. We view (2.5) as a generating function for all pairs (π, P) where $\pi \in S_g$ and P is any g-tuple of double lattice paths.

From Proposition 7 and Proposition 8, it follows that the generating function for all pairs $(\mathrm{id}, \mathsf{P})$ when P is separable, equals the generating function for all pairs $(P, \{a_i\}_{i=1}^r)$ where P is a non-crossing g-tuple of double lattice paths, that is,

$$(2.6) \qquad (\sum_{i=1}^\infty x_i)^r s_{\lambda/\mu}(X) = p_{1^r}(X) s_{\lambda/\mu}(X).$$

So in order to prove Theorem 2, it remains to find an involution on all pairs $(\pi, \mathcal{P})$ when $\mathcal{P}$ is not separable. By Definition 2.3, a g-tuple $\mathcal{P}$ of double lattice paths is *not separable* if one of conditions $(1), (2)$ in Definition 2.3 is not true.

It should be mentioned that conditions (1) and (2) are *disjoint*, that is, there is no common c such that neither (1) nor (2) is true. So we first consider the *minimal* integer $c_{\min}$ dissatisfying one of conditions $(1), (2)$ in Definition 2.3. Second we choose a *minimum* of any non-separable g-tuple

$$(2.7) \qquad \mathcal{P} = (\mathcal{P}(u_{\pi_1}, v_1), \mathcal{P}(u_{\pi_1}, v_1), \ldots, \mathcal{P}(u_{\pi_g}, v_g))$$

of double lattice paths to be

- a point $(c_{\min}, y)$ if it is the first point on line $x = c_{\min}$ from top to bottom such that two double lattice paths in $\mathcal{P}$ are intersecting at point $(c_{\min}, y)$.
- a pair $(c_{\min}, (y_1, y_2))$ where $y_1 < y_2$ if $(c_{\min}, y_2), (c_{\min}, y_1)$ are the first two points on line $x = c_{\min}$ from top to bottom such that there is a pair $(c_{\min}, [i, j])$ dissatisfying condition (2) of Definition 2.3, and $(c_{\min}, y_2), (c_{\min}, y_1)$ are respectively the $c_{\min}$-points of $\mathsf{P}(u_{\pi_j}, v_j), \mathsf{P}(u_{\pi_i}, v_i)$.

Example 14. *In Example 13 we have discussed the non-separable triple of double lattice paths in Fig. 2.6. In this case, $c_{\min} = 0$ and the minimum is $(c_{\min}, (y_1, y_2)) = (0, (3, 5))$.*

We are now ready to construct the involution f on all non-separable g-tuples $\mathcal{P}$ of double lattice paths by distinguishing the cases when the minimum of $\mathcal{P}$ is a point $(c_{\min}, y)$ or a pair $(c_{\min}, (y_1, y_2))$. For each case, we will express the involution f as

$$(\pi, \mathcal{P}) \mapsto (\sigma, \mathcal{Q}) = f((\pi, \mathcal{P}))$$

where $\pi, \sigma \in S_g$ and $\mathcal{P}, \mathcal{Q}$ are two non-separable g-tuples of double lattice paths with $\mathcal{P}$ given in (2.7) and with

$$(2.8) \qquad \mathcal{Q} = (\mathcal{Q}(u_{\sigma_1}, v_1), \mathcal{Q}(u_{\sigma_2}, v_2), \ldots, \mathcal{Q}(u_{\sigma_g}, v_g)).$$

For each case below, the involution f has the following properties:

(1) f is weight-preserving, that is, $\prod_{q=1}^{g} w(\mathcal{Q}(u_{\sigma_q}, v_q)) = \prod_{q=1}^{g} w(\mathcal{P}(u_{\pi_q}, v_q))$;
(2) f is sign-reversing, that is, $\mathrm{inv}(\pi) = \mathrm{inv}(\sigma) \pm 1$;
(3) f is closed, that is, $\mathcal{P}$ and $\mathcal{Q}$ belong to the same case (or subcase).

Case 1: if the minimum of $\mathcal{P}$ is the point $v := (c_{\min}, y)$, assume that $\mathcal{P}(u_{\pi_i}, v_i)$ and $\mathcal{P}(u_{\pi_j}, v_j)$ are two double lattice paths whose $c_{\min}$-points are the topmost and the second topmost among all double lattice paths in $\mathcal{P}$ that are passing point v.

Since neither $c_{\min}$ nor $c_{\min} - 1$ is the content of some common special corner of Φ, all steps of $\mathcal{P}$ between lines $x = c_{\min}$ and $x = c_{\min} + 1$ are all horizontal steps or all diagonal steps. Using the notations $\mathcal{P}(u_{\pi_i}, v)$ and $\mathcal{P}(v, v_i)$ to denote the segments of the double lattice path $\mathcal{P}(u_{\pi_i}, v_i)$ from u_{π_i} to v and from v to v_i (similarly for $\mathcal{P}(u_{\pi_j}, v_j)$), we may define the pair $(\sigma, \mathcal{Q}) = f((\pi, \mathcal{P}))$ where $\sigma = \pi \circ (i\,j)$ as follows. For $q \neq i$, $q \neq j$, we set $\mathcal{Q}(u_{\sigma_q}, v_q) = \mathcal{P}(u_{\pi_q}, v_q)$ and

$$\mathcal{Q}(u_{\sigma_i}, v_i) = \mathcal{P}(u_{\pi_j}, v)\mathcal{P}(v, v_i), \quad \mathcal{Q}(u_{\sigma_j}, v_j) = \mathcal{P}(u_{\pi_i}, v)\mathcal{P}(v, v_j).$$

We will show that $\mathcal{Q}(u_{\sigma_i}, v_i)$ is a double lattice path from $u_{\sigma_i} = u_{\pi_j}$ to v_i and $\mathcal{Q}(u_{\sigma_j}, v_j)$ is a double lattice path from $u_{\pi_i} = u_{\sigma_j}$ to v_j by discussing the ending points of the non-vertical steps between lines $x = c_{\min} - 1$ and $x = c_{\min} + 1$.

Here, without loss of generality, we assume that the steps between lines $x = c_{\min} - 1$ and $x = c_{\min}$ are horizontal, while the steps between lines $x = c_{\min}$ and $x = c_{\min} + 1$ are diagonal. Suppose that the ending points of non-vertical steps from $\mathcal{P}(u_{\pi_i}, v_i)$ are points $(c_{\min}, a_1)$ and $(c_{\min} + 1, a_2)$ where $a_1 > a_2$, and the ones from $\mathcal{P}(u_{\pi_j}, v_j)$ are points $v = (c_{\min}, y)$ and $(c_{\min} + 1, b_2)$ where $y > b_2$; see Fig. 2.7. Since $\mathcal{P}(u_{\pi_i}, v_i)$ and $\mathcal{P}(u_{\pi_j}, v_j)$ are intersecting at point $v = (c_{\min}, y)$, one has $a_2 < y < a_1$, which implies $b_2 < y < a_1$.

So there is no single up-vertical step on line $x = c_{\min}$ that is preceding the diagonal step in $\mathcal{Q}(u_{\sigma_i}, v_i)$ or $\mathcal{Q}(u_{\sigma_j}, v_j)$. This indicates that $\mathcal{Q}(u_{\sigma_i}, v_i)$ and $\mathcal{Q}(u_{\sigma_j}, v_j)$ are double lattice paths according to Definition 2.1.

Furthermore, f is closed within all non-separable g-tuples of double lattice paths that belong to case 1, because by construction the minimum of $\mathcal{Q}$ is also $v = (c_{\min}, y)$. See Fig. 2.7.
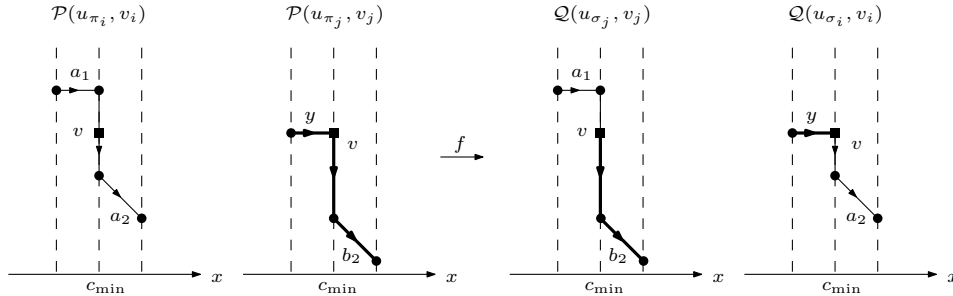
FIGURE 2.7. The involution f for case 1 when the minimum of $\mathcal{P}$ is a point $v = (c_{\min}, y)$ (marked by a black square) and a_1, a_2, y, b_2 represent the y-th coordinates of all ending points from non-vertical steps.

Case 2: if the minimum of $\mathcal{P}$ is a pair $(c_{\min}, (y_1, y_2))$ where $y_1 < y_2$, by assumption the triple $(c_{\min}, [i, j])$ fails to satisfy condition (2) of Definition 2.3, that is, for $c = c_{\min}$, neither $\mathcal{P}(u_{\pi_j}, v_j)$ after step (b), nor $\mathcal{P}(u_{\pi_i}, v_i)$ after step (d), is a double lattice path.

Suppose that between lines $x = c_{\min} - 1$ and $x = c_{\min} + 2$, the diagonal steps and the horizontal steps of $\mathcal{P}(u_{\pi_i}, v_i)$ and $\mathcal{P}(u_{\pi_j}, v_j)$ are given in Fig. 2.8, where all labels represent the y-th coordinates of all ending points from non-vertical steps. Note that y_1, y_3 have no value relation, that is $y_1 \le y_3$ or $y_1 > y_3$. So are y_2 and y_4. It should be mentioned that the
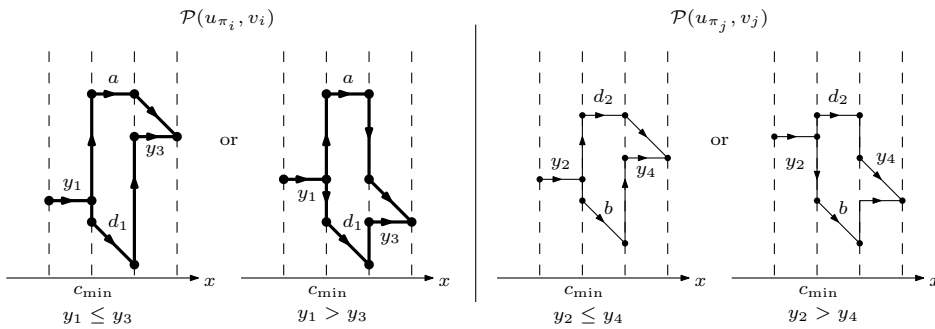

FIGURE 2.8. The steps of $\mathcal{P}(u_{\pi_i}, v_i)$ and $\mathcal{P}(u_{\pi_j}, v_j)$ between lines $x = c_{\min} - 1$ and $x = c_{\min} + 2$ if the minimum of $\mathcal{P}$ is a pair $(c_{\min}, (y_1, y_2))$.

horizontal step ending at $(c_{\min} + 1, d_2)$ and the diagonal step ending at $(c_{\min} + 2, y_4)$ are not contained in $\mathcal{P}(u_{\pi_j}, v_j)$ if the starting box or the ending box of Θ_j is the common special corner of Θ_i and Θ_j. But for this situation the discussion on the involution f follows

analogously, so we focus on the case when these two steps are contained in $\mathcal{P}(u_{\pi_j}, v_j)$. Likewise, we focus on the case when the diagonal step ending at $(c_{\min} + 1, d_1)$ and the horizontal step ending at $(c_{\min} + 2, y_3)$ are contained in $\mathcal{P}(u_{\pi_i}, v_i)$.

Note that we can identify any double lattice path $\mathcal{P}(u_{\pi_m}, v_m)$ as a pair

$$\mathcal{P}(u_{\pi_m}, v_m) = (\mathcal{P}_+(u_{\pi_m}, v_m), \mathcal{P}_-(u_{\pi_m}, v_m))$$

of lattice paths where all steps in $\mathcal{P}_+(u_{\pi_m}, v_m)$ are above or the same as the ones in $\mathcal{P}_-(u_{\pi_m}, v_m)$. For example, in Fig. 2.8, the horizontal step ending at $(c_{\min}, y_1)$, the horizontal step ending at $(c_{\min} + 1, a)$, the diagonal step ending at $(c_{\min} + 2, y_3)$, and all vertical steps in between belong to $\mathcal{P}_+(u_{\pi_i}, v_i)$. The horizontal step ending at $(c_{\min}, y_1)$, the diagonal step ending at $(c_{\min} + 1, d_1)$, the horizontal step ending at $(c_{\min} + 2, y_3)$, and all vertical steps in between belong to $\mathcal{P}_-(u_{\pi_i}, v_i)$.

Since neither $\mathcal{P}(u_{\pi_j}, v_j)$ after step (b) nor $\mathcal{P}(u_{\pi_i}, v_i)$ after step (d) is a double lattice path, the integer a must satisfy $a \geq y_2$ or $a > y_4$ and the integer b must satisfy $b < y_1$ or $b \leq y_3$. In other words, $\mathcal{P}_+(u_{\pi_i}, v_i), \mathcal{P}_+(u_{\pi_j}, v_j)$ are intersecting and $\mathcal{P}_-(u_{\pi_i}, v_i), \mathcal{P}_-(u_{\pi_j}, v_j)$ are intersecting on lines $x = c_{\min}$ or $x = c_{\min} + 1$.

Let w_1, w_2 be the intersecting points, respectively, of $\mathcal{P}_+(u_{\pi_i}, v_i), \mathcal{P}_+(u_{\pi_j}, v_j)$ and of $\mathcal{P}_-(u_{\pi_i}, v_i), \mathcal{P}_-(u_{\pi_j}, v_j)$. Then, as in case 1, we will switch the segments after points w_1, w_2 between $\mathcal{P}(u_{\pi_i}, v_i)$ and $\mathcal{P}(u_{\pi_j}, v_j)$.

Using $\mathcal{P}_+(u_{\pi_m}, w_1)$ and $\mathcal{P}_+(w_1, v_m)$ to denote the segments of $\mathcal{P}_+(u_{\pi_m}, v_m)$ from u_{π_m} to w_1, and from w_1 to v_m, respectively. Similarly for $\mathcal{P}_-(u_{\pi_m}, w_2)$ and $\mathcal{P}_-(w_2, v_m)$. We may define the pair $(\sigma, \mathcal{Q}) = f((\pi, \mathcal{P}))$ where $\sigma = \pi \circ (i\, j)$ as follows. For $q \neq i, q \neq j$, we set $\mathcal{Q}(u_{\sigma_q}, v_q) = \mathcal{P}(u_{\pi_q}, v_q)$. For $q = i$ or $q = j$, $\mathcal{Q}(u_{\sigma_q}, v_q) = (\mathcal{Q}_+(u_{\sigma_q}, v_q), \mathcal{Q}_-(u_{\sigma_q}, v_q))$ where

$$\mathcal{Q}_+(u_{\sigma_i}, v_i) = \mathcal{P}_+(u_{\pi_j}, w_1)\mathcal{P}_+(w_1, v_i), \quad \mathcal{Q}_-(u_{\sigma_i}, v_i) = \mathcal{P}_-(u_{\pi_j}, w_2)\mathcal{P}_-(w_2, v_i).$$
$$\mathcal{Q}_+(u_{\sigma_j}, v_j) = \mathcal{P}_+(u_{\pi_i}, w_1)\mathcal{P}_+(w_1, v_j), \quad \mathcal{Q}_-(u_{\sigma_j}, v_j) = \mathcal{P}_-(u_{\pi_i}, w_2)\mathcal{P}_-(w_2, v_j).$$

We need to show that $\mathcal{Q}(u_{\sigma_i}, v_i)$ is a double lattice path from $u_{\sigma_i} = u_{\pi_j}$ to v_i and $\mathcal{Q}(u_{\sigma_j}, v_j)$ is a double lattice path from $u_{\sigma_j} = u_{\pi_i}$ to v_j, which is similar to the proof of case 1. By discussing the locations of the intersecting points w_1, w_2, we could discuss the following disjoint sub-cases:

> case 2.1: w_1 and w_2 are both on line $x = c_{\min}$;
> case 2.2: w_1 is on line $x = c_{\min} + 1$ and w_2 is on line $x = c_{\min}$;
> case 2.3: w_1 is on line $x = c_{\min}$ and w_2 is on line $x = c_{\min} + 1$;
> case 2.4: w_1 and w_2 are both on line $x = c_{\min} + 1$.

Case 2.1: We observe that w_1 and w_2 are both on line $x = c_{\min}$ if and only if $b < y_1 < y_2 \leq a$. In this case $w_1 = (c_{\min}, y_2)$ and $w_2 = (c_{\min}, y_1)$; see Fig. 2.9. Since $d_1 < y_1 < y_2 \leq a$ and $d_2 > y_2 > y_1 > b$, $\mathcal{Q}(u_{\sigma_i}, v_i)$ and $\mathcal{Q}(u_{\sigma_j}, v_j)$ are double lattice paths.

Case 2.2: We observe that w_1 is on line $x = c_{\min} + 1$ and w_2 is on line $x = c_{\min}$ if and only if $b < y_1$ and $y_4 < a < y_2$. In this case $w_1 = (c_{\min} + 1, a)$ and $w_2 = (c_{\min}, y_1)$; see Fig. 2.10. Note that $\mathcal{Q}(u_{\sigma_i}, v_i)$ and $\mathcal{Q}(u_{\sigma_j}, v_j)$ are double lattice paths. This is guaranteed by the assumption $b < y_1$ and $y_4 < a < y_2$. To be precise, $d_1 < y_2$ holds because $d_1 < y_1$
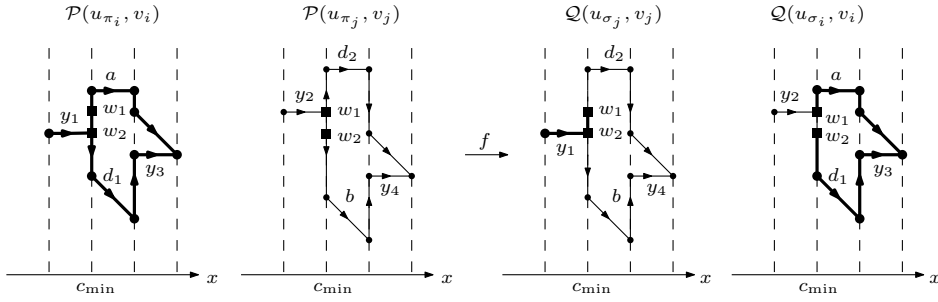
FIGURE 2.9. The involution f for case 2.1 when the minimal of $\mathcal{P}$ is a pair $(c_{\min}, (y_1, y_2))$ and both w_1, w_2 are on line $x = c_{\min}$.

and $y_1 < y_2$; $y_3 < d_2$ holds because $y_3 < a$ and $a < y_2 < d_2$; $b < y_1$ and $a > y_4$ hold because of the assumption.
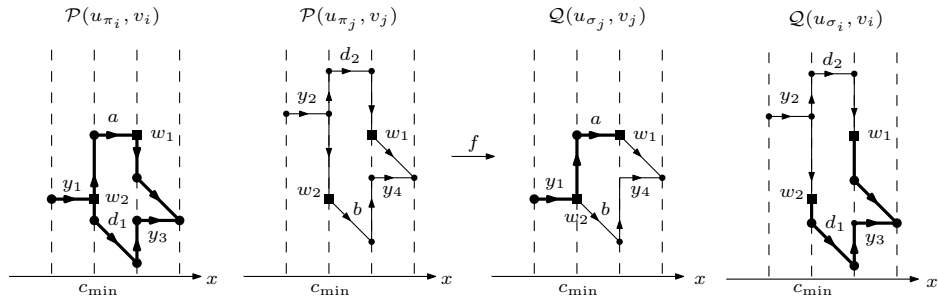

FIGURE 2.10. The involution f for case 2.2 when the minimal of $\mathcal{P}$ is a pair $(c_{\min}, (y_1, y_2))$ and w_1 is on line $x = c_{\min} + 1$, w_2 is on line $x = c_{\min}$.

Case 2.3: We observe that w_1 is on line $x = c_{\min}$ and w_2 is on line $x = c_{\min} + 1$ if and only if $y_1 \le b \le y_3$ and $a \ge y_2$. In this case $w_1 = (c_{\min}, y_2)$ and $w_2 = (c_{\min} + 1, b)$; see Fig. 2.11. Note that $\mathcal{Q}(u_{\sigma_i}, v_i)$ and $\mathcal{Q}(u_{\sigma_j}, v_j)$ are double lattice paths. This is guaranteed by the assumption $y_1 \le b \le y_3$ and $a \ge y_2$. To be precise, $d_2 > y_1$ holds because $d_2 > y_2$ and $y_2 > y_1$; $d_1 \le y_4$ holds because $d_1 < y_1 \le b \le y_4$; $a \ge y_2$ and $b \le y_3$ hold because of the assumption.

Case 2.4: We observe that both w_1, w_2 are on line $x = c_{\min} + 1$ if and only if $y_1 \le b \le y_3$ and $y_4 < a < y_2$. In this case $w_1 = (c_{\min} + 1, a)$ and $w_2 = (c_{\min} + 1, b)$; see Fig. 2.12. Note that $\mathcal{Q}(u_{\sigma_i}, v_i)$ and $\mathcal{Q}(u_{\sigma_j}, v_j)$ are double lattice paths. This is guaranteed by the assumption $y_1 \le b \le y_3$ and $y_4 < a < y_2$. To be precise, $d_1 < y_4 < a$ holds because $d_1 < y_1 \le b \le y_4 < a$; and $b \le y_3 < d_2$ holds because $b \le y_3 < a < y_2 \le d_2$.

For each case (case 1 or case 2.1-2.4), it is clear that $(\pi, \mathcal{P}) \mapsto (\sigma, \mathcal{Q}) = f((\pi, \mathcal{P}))$ is an involution which preserves the weight of the double lattice path and changes the inversion of the permutation by 1. Furthermore, f is closed within all non-separable g-tuples of double lattice paths that belong to the same case, because two intersecting points w_1, w_2
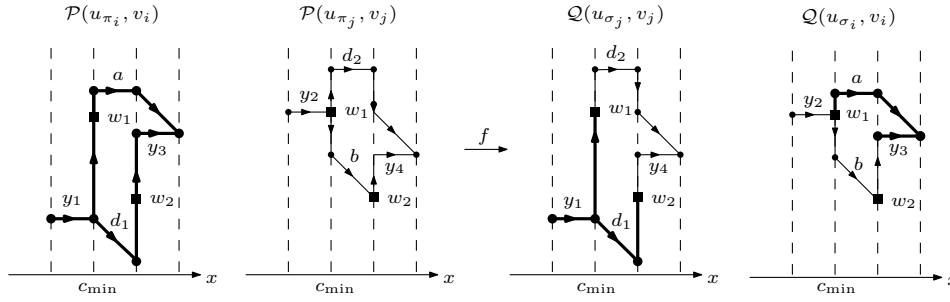
FIGURE 2.11. The involution f for case 2.3 when the minimal of $\mathcal{P}$ is a pair $(c_{\min}, (y_1, y_2))$ and w_1 is on line $x = c_{\min}$, w_2 is on line $x = c_{\min} + 1$.
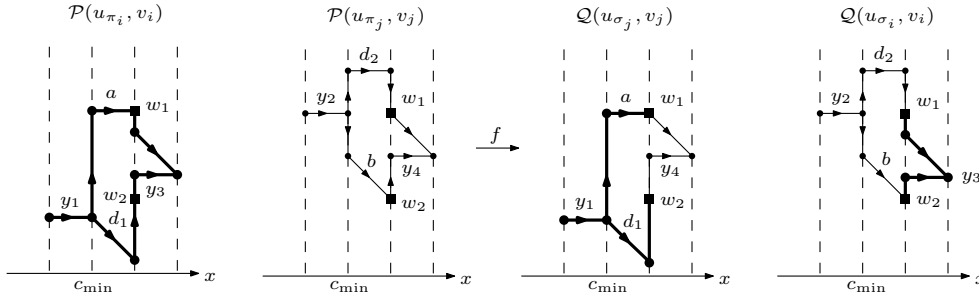

FIGURE 2.12. The involution f for case 2.4 when the minimal of $\mathcal{P}$ is a pair $(c_{\min}, (y_1, y_2))$ and w_1, w_2 are on line $x = c_{\min} + 1$.

are fixed with respect to f and the minimum of $\mathcal{Q}$ is also the pair $(c_{\min}, (y_1, y_2))$; see Fig. 2.9-2.12.

Example 15. *In example 13 we have discussed the non-separable double lattice paths in Fig. 2.6. In this case, $c_{\min} = 0$ and the minimum is $(0, (3, 5))$. Two double lattice paths $\mathcal{P}(u_3, v_2), \mathcal{P}(u_2, v_1)$, after the involution f, are $\mathcal{Q}(u_2, v_2), \mathcal{Q}(u_3, v_1)$ shown in Fig. 2.13, while $\mathcal{Q}(u_1, v_3) = \mathcal{P}(u_1, v_3)$. We mark two intersecting points w_1, w_2 in Fig. 2.13 and both $\mathcal{P}, \mathcal{Q}$ belong to case 2.3.*

Proof of Theorem 2. From (2.5) and the involution $f : (\pi, \mathcal{P}) \mapsto (\sigma, \mathcal{Q})$ in subsection 2.3, we find that only the generating function for all pairs $(\mathrm{id}, \mathsf{P})$ where P is any separable g-tuple of double lattice paths, is remained on the right hand side of (2.5). In combination of (2.6), (1.2) follows immediately.

Proof of Corollary 3. We refer the readers to Chapter 7 of [13] for a full description of the exponential specialization. Let $[x_1 x_2 \cdots x_n] f$ denote the coefficient of $x_1 x_2 \cdots x_n$ in f. Then the exponential specialization ex of the symmetric function f is defined as

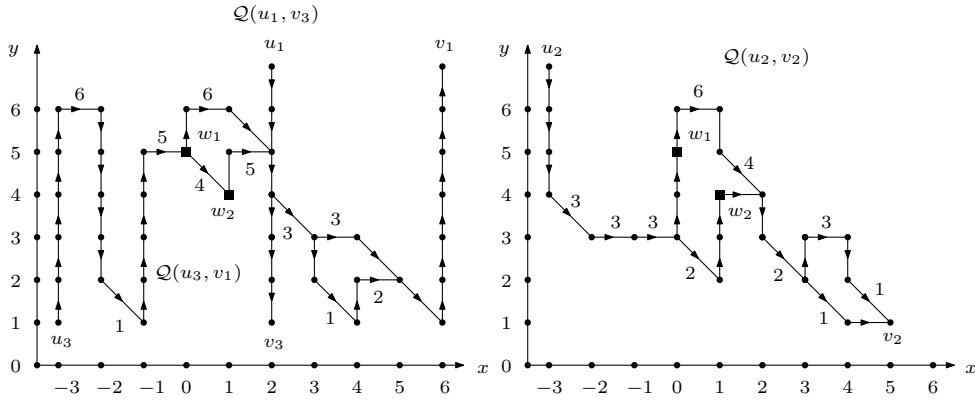$$\mathrm{ex}(f) = \sum_{n \geq 0} [x_1 x_2 \cdots x_n] f \frac{t^n}{n!}$$

FIGURE 2.13. A non-separable triple $\mathcal{Q}$ of double lattice paths with $\sigma = 321$ that uniquely corresponds to the pair $(\pi, \mathcal{P})$ under the involution f, where $\mathcal{P}$ is given in Fig. 2.6 and $\pi = 231$.

and $\mathrm{ex}_1(f) = \mathrm{ex}(f)_{t=1}$. Let $N = |\lambda/\mu|$ and $a_{i,j} = |\Theta_i \# \Theta_j|$, then one has

$$\mathrm{ex}(p_{1^r}(X)) = t^r, \ \ \mathrm{ex}(s_{\lambda/\mu}(X)) = \frac{f^{\lambda/\mu}}{N!} t^N \ \ \text{and} \ \ \mathrm{ex}(s_{\Theta_i \# \Theta_j}(X)) = \frac{f^{\Theta_i \# \Theta_j}}{a_{i,j}!} t^{a_{i,j}}.$$

Consequently (1.3) follows directly after we apply ex_1 on both sides of (1.2).

3. APPLICATION TO THE ENUMERATION OF m-STRIP TABLEAUX

We will count the number of m-strip tableaux by applying Corollary 3. It should be pointed out that the enumeration of $2k$-strip tableaux is a direct consequence of Theorem 1; see [8]. In [12], Morales, Pak and Panova also found that the enumeration of $2k$-strip tableaux can be simplified by applying Lascoux-Pragacz's theorem [10], or more generally, Hamel and Goulden's theorem (Theorem 1).

3.1. **The m-strip tableaux.** Baryshnikov and Romik [2] counted the number of m-strip tableaux as a generalization of the classical formula from D. André [1] on the number of up-down permutations.

Definition 3.1 (m-strip tableaux). An *m-strip diagram* $\mathcal{D}_m(\tilde{\lambda}; \tilde{\mu})$ contains three parts: head $\tilde{\lambda}$, tail $\tilde{\mu}$ and body. The body of an m-strip diagram consists of an elongated hexagonal shape with n columns, where the numbers of boxes in the n columns are

$$\underbrace{\left\lceil \frac{m+1}{2} \right\rceil, \left\lceil \frac{m+1}{2} \right\rceil + 1, \ldots, m-1, m}_{\text{length: } \lfloor \frac{m-1}{2} \rfloor}, m, \ldots, m, \underbrace{m-1, \ldots, \left\lceil \frac{m+1}{2} \right\rceil + 1, \left\lceil \frac{m+1}{2} \right\rceil}_{\text{length: } \lfloor \frac{m-1}{2} \rfloor}.$$

The last $\lfloor (m-1)/2 \rfloor$ columns forms a standard diagram, while the first $\lfloor (m-1)/2 \rfloor$ columns, after rotating by 180 degrees in the plane, forms a standard diagram. The columns where

each contains m boxes forms a skew diagram of shape

$$\underbrace{(n - 2 \left\lfloor \frac{m-1}{2} \right\rfloor, \ldots, n - 2 \left\lfloor \frac{m-1}{2} \right\rfloor}_{\text{length: } m}, n - 2 \left\lfloor \frac{m-1}{2} \right\rfloor - 1, \ldots, 2, 1)/(n - 2 \left\lfloor \frac{m-1}{2} \right\rfloor - 1, \ldots, 2, 1).$$

The *head* $\tilde{\lambda}$ and *tail* $\tilde{\mu}$ are standard diagrams of length at most $\lfloor m/2 \rfloor$ that are rotated and connected to the body by leaning against the sides of the body. The empty partition (0) is always denoted by $\varnothing$ and *an m-strip tableau* is a standard Young tableau of the m-strip shape.

Remark 3.1. *Our definition of m-strip diagrams is slightly different to the one in [2] because [2] contains a minor typo on the number of boxes in the leftmost and the rightmost columns of any m-strip diagram. Our notation $\mathcal{D}_m(\tilde{\lambda}; \tilde{\mu})$ is the notation D in [2] and we find it more convenient to use $\mathcal{D}_m(\tilde{\lambda}; \tilde{\mu})$ to represent some m-strip diagrams for small m.*

Example 16. *See Fig. 3.1 for an example of 6-strip diagram with head partition $\tilde{\lambda}$ and tail partition $\tilde{\mu}$ from [2]. The standard diagrams of $\tilde{\lambda} = (3, 1), \tilde{\mu} = (2, 2)$ are rotated and attached to the body of the 6-strip diagram.*
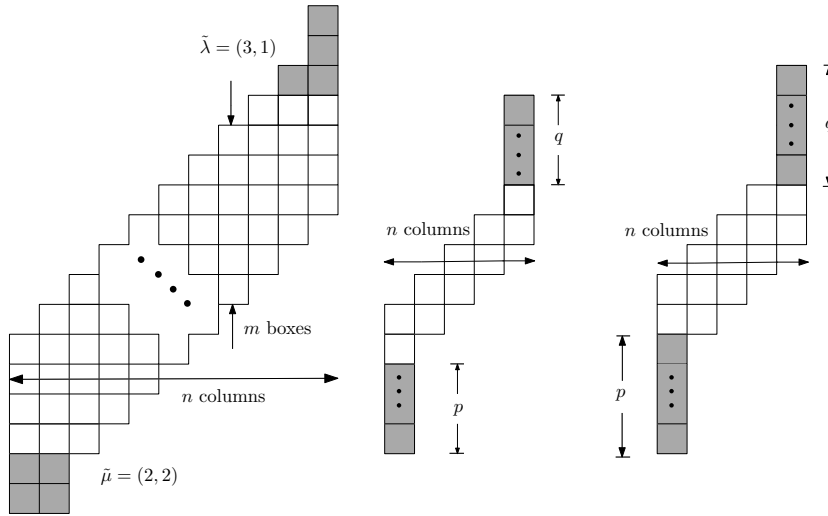

FIGURE 3.1. A 6-strip diagram $\mathcal{D}_6(\tilde{\lambda}; \tilde{\mu})$, a 2-strip diagram $\mathcal{D}_2((q); (p))$ and a 3-strip diagram $\mathcal{D}_3((q); (p))$ (left, middle, right).

To avoid confusion, we adopt the definitions and notations of Euler numbers and tangent numbers from [2]. A permutation $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$ is called an *up-down permutation* if $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots$. It is well-known that the exponential generating function of the numbers A_n of up-down permutations of $[n] = \{1, 2, \ldots, n\}$ is

$$(3.1) \qquad \qquad \sum_{n \geq 0} A_n \frac{x^n}{n!} = \sec x + \tan x.$$

This is also called André's theorem [1], which connects the numbers A_n with the *Euler numbers* E_n and *tangent numbers* T_n by the Taylor expansions of $\sec x$ and $\tan x$, that is,

$$\sec x = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n} x^{2n}}{(2n)!} \quad \text{and} \quad \tan x = \sum_{n=1}^{\infty} \frac{T_n x^{2n-1}}{(2n-1)!}.$$

This implies that

$$(3.2) \qquad\qquad A_{2n} = (-1)^n E_{2n} \quad \text{and} \quad A_{2n-1} = T_n.$$

It should be mentioned that Euler numbers are defined differently in some literature [10, 13].

It is clear that an up-down permutation of $[2n]$ can be identified as a 2-strip tableau of shape $\mathcal{D}_2(\varnothing; \varnothing)$. By thickening the 2-strip diagram, Baryshnikov and Romik [2] introduced the m-strip diagram and enumerated the m-strip tableaux via transfer operators, which proved that the determinant to count m-strip tableaux has order $\lfloor m/2 \rfloor$. This is certainly to their advantage that Baryshnikov and Romik's determinant for $(2k+1)$-strip diagrams is much simpler than the one directly from Hamel and Goulden's theorem (Theorem 1). We next recall the Baryshnikov and Romik's determinant for the m-strip tableaux. We define the numbers

$$(3.3) \qquad\qquad \bar{A}_n = \frac{A_n}{n!}, \ \tilde{A}_n = \frac{\bar{A}_n}{2^{n+1} - 1} \ \text{and} \ \hat{A}_n = \frac{(2^n - 1)\bar{A}_n}{2^n(2^{n+1} - 1)},$$

and denote the head Young diagram by $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_k)$ and the tail Young diagram by $\tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_k)$ where $k = \lfloor m/2 \rfloor$. For any non-negative integers p, q, we denote by $\alpha_{n,2}(p,q)$ and $\alpha_{n,3}(p,q)$ the number of 2-strip tableaux of shape $\mathcal{D}_2((q); (p))$ and the number of 3-strip tableaux of shape $\mathcal{D}_3((q); (p))$ where the empty partition (0) is denoted by $\varnothing$. In other words,

$$\alpha_{n,2}(p,q) = f^{\mathcal{D}_2((q);(p))} \quad \text{and} \quad \alpha_{n,3}(p,q) = f^{\mathcal{D}_3((q);(p))}.$$

In particular, $\alpha_{n,2}(0,0) = A_{2n}$ and some values of $\alpha_{n,3}(p,q)$ are given in Theorem 10. Note that $\alpha_{n,2}(p,q) = \alpha_{n,2}(q,p)$ holds for any non-negative integers p and q. This is true because for any standard Young tableau T of shape $\mathcal{D}_2((q); (p))$, if we replace every entry w of T by $2n + p + q + 1 - w$ and flip the diagram $\mathcal{D}_2((q); (p))$ upside-down and reverse it left-to-right, we obtain a standard Young tableau of shape $\mathcal{D}_2((p); (q))$. Similarly $\alpha_{n,3}(p,q) = \alpha_{n,3}(q,p)$ holds for any non-negative integers p and q. Furthermore, we define the numbers $X_{2n-1}(p,q)$ and $Y_{2n-2}(p,q)$ as below:

$$X_{2n-1}(p,q) = \frac{\alpha_{n,2}(p,q)}{(2n+p+q)!} \quad \text{and} \quad Y_{2n-2}(p,q) = \frac{\alpha_{n,3}(p,q)}{(3n+p+q-2)!}.$$

The numbers $X_{2n-1}(p,q), Y_{2n-2}(p,q)$ are the same as the ones in [2], while our notation $\alpha_{n,2}(p,q)$ is α_n in [2], because we need the parameters p, q to describe the thickened strips later. For the readers' convenience, we should mention that the left 2-strip diagram in Fig. 4 of [2] should be the middle one in Fig. 3.1. Baryshnikov and Romik proved that

Theorem 9 ([2]). *Let $L_i = \tilde{\lambda}_i + k - i$ and $M_i = \tilde{\mu}_i + k - i$ for $1 \leq i \leq k = \lfloor m/2 \rfloor$. Then the number of standard Young tableaux of shape $\mathcal{D}_m(\tilde{\lambda}; \tilde{\mu})$ is given by*

$$(3.4) \qquad f^{\mathcal{D}_m(\tilde{\lambda};\tilde{\mu})} = (-1)^{\binom{k}{2}}|\mathcal{D}_m(\tilde{\lambda};\tilde{\mu})|! \det[X_{2n-m+1}(L_i, M_j)]_{i,j=1}^k \ \text{ if } m = 2k \ \text{ or by}$$

$$(3.5) \qquad f^{\mathcal{D}_m(\tilde{\lambda};\tilde{\mu})} = (-1)^{\binom{k}{2}}|\mathcal{D}_m(\tilde{\lambda};\tilde{\mu})|! \det[Y_{2n-m+1}(L_i, M_j)]_{i,j=1}^k \ \text{ if } m = 2k+1.$$

Remark 3.2. *Theorem 9 is a combination of Theorem 4 and 5 in [2]. Here we use the combinatorial interpretations of $\alpha_{n,2}(p,q), \alpha_{n,3}(p,q)$ to introduce the numbers $X_{2n-1}(p,q), Y_{2n-2}(p,q)$, whose expressions in terms of the numbers $\hat{A}_i$, $\tilde{A}_i$ and $\bar{A}_i$ can be derived by the recursions of $\alpha_{n,2}(p,q)$ and $\alpha_{n,3}(p,q)$. Here we omit the computational details.*

Baryshnikov and Romik [2] also presented some explicit formulas for small m. We will establish Theorem 10 by decomposing 3-strip tableaux directly and by choosing two different outside nested decompositions respectively for $4, 5$-strip tableaux.

Theorem 10 ([2]). *Some numbers of 3-strip tableaux are*

$$(3.6) \qquad \alpha_{n,3}(0,0) = f^{\mathcal{D}_3(\varnothing;\varnothing)} = \frac{(3n-2)!T_n}{(2n-1)!2^{2n-2}} = \frac{(3n-2)!\bar{A}_{2n-1}}{2^{2n-2}},$$

$$(3.7) \qquad \alpha_{n,3}(0,1) = f^{\mathcal{D}_3((1);\varnothing)} = \frac{(3n-1)!T_n}{(2n-1)!2^{2n-1}} = \frac{(3n-1)!\bar{A}_{2n-1}}{2^{2n-1}},$$

$$(3.8) \qquad \alpha_{n,3}(1,1) = f^{\mathcal{D}_3((1);(1))} = \frac{(3n)!(2^{2n-1}-1)T_n}{(2n-1)!2^{2n-1}(2^{2n}-1)} = (3n)!\hat{A}_{2n-1}.$$

Some numbers of 4-strip tableaux are

$$(3.9) \quad f^{\mathcal{D}_4(\varnothing;\varnothing)} = \binom{4n-2}{2n-1}T_n^2 + \binom{4n-2}{2n-2}E_{2n-2}E_{2n} = (4n-2)!\det\begin{bmatrix} \bar{A}_{2n-1} & \bar{A}_{2n} \\ \bar{A}_{2n-2} & \bar{A}_{2n-1} \end{bmatrix},$$

$$(3.10)$$

$$f^{\mathcal{D}_4((1);(1))} = \binom{4n}{2n}E_{2n}^2 - \binom{4n}{2n-2}E_{2n-2}E_{2n+2} = (4n)!\det\begin{bmatrix} \bar{A}_{2n} & \bar{A}_{2n+2} \\ \bar{A}_{2n-2} & \bar{A}_{2n} \end{bmatrix},$$

and the number of 5-strip tableaux without head and tail is

$$(3.11) \qquad f^{\mathcal{D}_5(\varnothing;\varnothing)} = \frac{(5n-6)!\, T_{n-1}^2}{((2n-3)!)^2\, 2^{4n-6}(2^{2n-2}-1)} = (5n-6)!\det\begin{bmatrix} \tilde{A}_{2n-3} & \hat{A}_{2n-3} \\ \hat{A}_{2n-3} & \tilde{A}_{2n-3} \end{bmatrix}.$$

3.2. **Proof of Theorem 9 and Theorem 10.**

3.2.1. *Proof of (3.4).* We count the number $f^{\mathcal{D}_{2k}(\tilde{\lambda};\tilde{\mu})}$ of $2k$-strip tableaux by choosing an outside decomposition $\phi = (\theta_1, \theta_2, \ldots, \theta_k)$ of the $2k$-strip diagram $\mathcal{D}_{2k}(\tilde{\lambda}; \tilde{\mu})$, which is a special outside nested decomposition without common special corners. Given a $2k$-strip diagram $\mathcal{D}_{2k}(\tilde{\lambda}; \tilde{\mu})$, we can peel this diagram off into successive maximal outer strips $\theta_1, \theta_2, \ldots, \theta_k$ beginning from the outside; see the left one in Fig. 3.2.

We recall the numbers $L_i = \tilde{\lambda}_i + k - i$ and $M_i = \tilde{\mu}_i + k - i$, for $1 \leq i \leq k$. In the outside decomposition ϕ, every strip θ_i is a 2-strip of $(n - k + 1)$ columns, with head
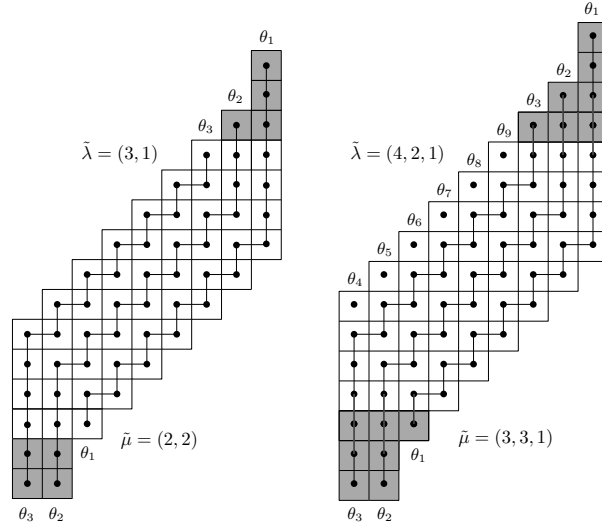
FIGURE 3.2. The outside nested decomposition $\phi = (\theta_1, \theta_2, \theta_3)$ that we choose for the 6-strip diagram $\mathcal{D}_6((3,1);(2,2))$ (left) and the outside nested decomposition that we will not choose for the 7-strip diagram $\mathcal{D}_7((4,2,1);(3,3,1))$ (right).

partition (L_i) and tail partition (M_{k-i+1}). The number of such tableaux are denoted by $\alpha_{n-k+1,2}(L_i, M_{k-i+1})$, that is, $f^{\theta_i} = \alpha_{n-k+1,2}(L_i, M_{k-i+1})$. By Definition 1.9, we see that the thickened cutting strip $H(\phi)$ is a 2-strip of $(n-k+1)$ columns, with head partition (L_1) and tail partition (M_1). So it follows that $\theta_i \# \theta_j$ is a 2-strip diagram with $(n-k+1)$ columns, with head partition (L_i) and with tail partition (M_{k-j+1}). Consequently, $f^{\theta_i \# \theta_j} = \alpha_{n-k+1,2}(M_{k-j+1}, L_i) = \alpha_{n-k+1,2}(L_i, M_{k-j+1})$. By Corollary 3 we know that the number $f^{\mathcal{D}_{2k}(\tilde{\lambda};\tilde{\mu})}$ of standard Young tableaux of $2k$-strip shape with n columns, is expressed as a determinant where the (i,j)-th entry is $\alpha_{n-k+1,2}(L_i, M_{k-j+1})/(2n-2k+L_i+M_{k-j+1}+2)! = X_{2n-2k+1}(L_i, M_{k-j+1})$. That is to say,

$$f^{\mathcal{D}_{2k}(\tilde{\lambda};\tilde{\mu})} = |\mathcal{D}_{2k}(\tilde{\lambda};\tilde{\mu})|! \det[X_{2n-2k+1}(L_i, M_{k-j+1})]_{i,j=1}^k$$
$$= (-1)^{\binom{k}{2}} |\mathcal{D}_{2k}(\tilde{\lambda};\tilde{\mu})|! \det[X_{2n-2k+1}(L_i, M_j)]_{i,j=1}^k,$$

which is (3.4).

3.2.2. *Proof of (3.5).* We observe that any outside decomposition of $(2k+1)$-strip diagram will not reduce the order of the Jacobi-Trudi determinant in Theorem 4 because the minimal number of strips contained in any outside decomposition is exactly the number of columns in any $(2k+1)$-strip diagram $\mathcal{D}_{2k+1}(\tilde{\lambda};\tilde{\mu})$; see the outside decomposition of the 7-strip diagram $\mathcal{D}_7((4,2,1);(3,3,1))$ in Fig. 3.2.

Given a $(2k+1)$-strip diagram $\mathcal{D}_{2k+1}(\tilde{\lambda};\tilde{\mu})$, we can peel this diagram off into successive maximal outer thickened strips $\Theta_1, \Theta_2, \ldots, \Theta_k$ beginning from the outside; see Fig. 3.3.
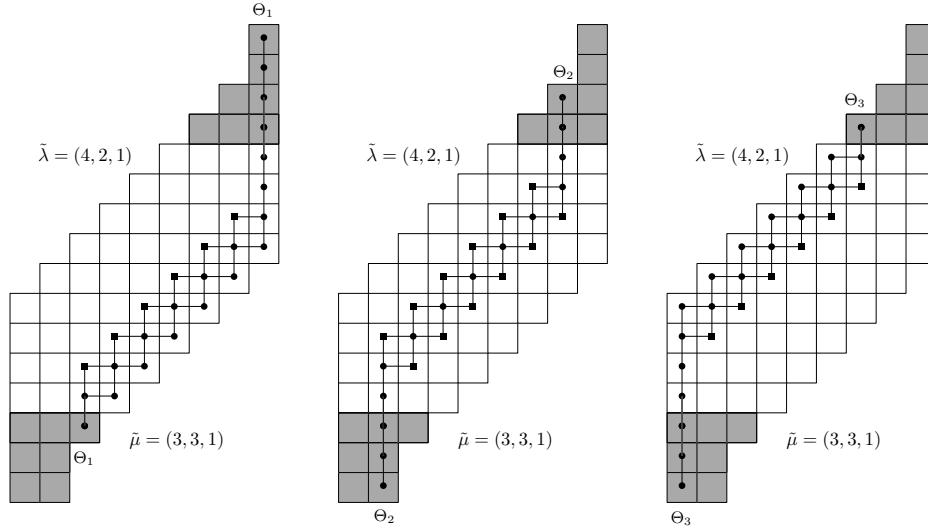
FIGURE 3.3. The outside nested decomposition $\Phi = (\Theta_1, \Theta_2, \Theta_3)$ of the 7-strip diagram $\mathcal{D}_7((4,2,1);(3,3,1))$ where each common special corner is marked by a black square in the thickened strip.

Consider the outside nested decomposition $\Phi = (\Theta_1, \Theta_2, \ldots, \Theta_k)$, every thickened strip Θ_i is a 3-strip of $(n-k+1)$ columns, with head partition (L_i) and tail partition (M_{k-i+1}). The number of such tableaux are denoted by $\alpha_{n-k+1,3}(L_i, M_{k-i+1})$, that is, $f^{\Theta_i} = \alpha_{n-k+1,3}(L_i, M_{k-i+1})$. By Definition 1.9, we see that the thickened cutting strip $H(\Phi)$ is a 3-strip of $(n-k+1)$ columns, with head partition (L_1) and tail partition (M_1). So it follows that $\Theta_i \# \Theta_j$ is a 3-strip diagram with $(n-k+1)$ columns, with head partition (L_i) and with tail partition (M_{k-j+1}). Consequently, $f^{\Theta_i \# \Theta_j} = \alpha_{n-k+1,3}(M_{k-j+1}, L_i) = \alpha_{n-k+1,3}(L_i, M_{k-j+1})$. By Corollary 3 we know that the number $f^{\mathcal{D}_{2k+1}(\tilde{\lambda};\tilde{\mu})}$ of standard Young tableaux of $(2k+1)$-strip shape with n columns, is expressed as a determinant where the (i,j)-th entry is $\alpha_{n-k+1,3}(L_i, M_{k-j+1})/(3n - 3k + L_i + M_{k-j+1} + 1)! = Y_{2n-2k}(L_i, M_{k-j+1})$. That is to say,

$$f^{\mathcal{D}_{2k+1}(\tilde{\lambda};\tilde{\mu})} = |\mathcal{D}_{2k+1}(\tilde{\lambda};\tilde{\mu})|! \det[Y_{2n-2k}(L_i, M_{k-j+1})]_{i,j=1}^{k}$$

$$= (-1)^{\binom{k}{2}} |\mathcal{D}_{2k+1}(\tilde{\lambda};\tilde{\mu})|! \det[Y_{2n-2k}(L_i, M_j)]_{i,j=1}^{k},$$

which is (3.5).

3.2.3. *Proof of (3.6)-(3.8).* Here we need the parameter n to describe the number of columns when we decompose the 3-strip diagrams. So we set

$$\mathcal{D}_{3n-2} = \mathcal{D}_3(\varnothing; \varnothing), \quad \mathcal{D}_{3n-1} = \mathcal{D}_3((1); \varnothing),$$
$$\mathcal{D}^*_{3n-1} = \mathcal{D}_3(\varnothing; (1)), \quad \mathcal{D}_{3n} = \mathcal{D}_3((1); (1)).$$

and let $\mathcal{C}_{3n}$ denote a 3-strip diagram which is obtained by adding a new box to the right of the topmost and rightmost box of $\mathcal{D}_3(\varnothing; (1))$; see Fig. 3.5. First we have two simple observations.

$$\mathcal{D}_{3n-2}=\mathcal{D}_3(\varnothing;\varnothing) \qquad \mathcal{D}^*_{3n-1}=\mathcal{D}_3(\varnothing;(1)) \qquad \mathcal{D}_{3n-1}=\mathcal{D}_3((1);\varnothing) \qquad \mathcal{D}_3((1);(1))=\mathcal{D}_{3n} \qquad \mathcal{C}_{3n}$$
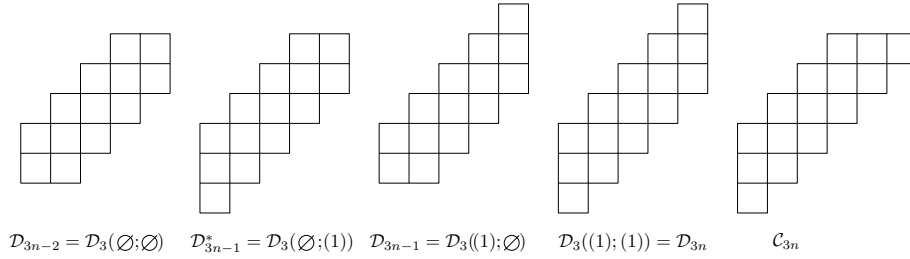
FIGURE 3.4. 3-strip diagrams

Lemma 11. *The numbers $f^{\mathcal{D}_{3n-2}}$, $f^{\mathcal{D}_{3n-1}}$ and $f^{\mathcal{D}_{3n}}$ satisfy*

$$(3.12) \qquad (3n-1)f^{\mathcal{D}_{3n-2}} = 2f^{\mathcal{D}_{3n-1}},$$

$$(3.13) \qquad (3n)f^{\mathcal{D}_{3n-1}} = f^{\mathcal{D}_{3n}} + f^{\mathcal{C}_{3n}}.$$

Proof. Let $\mathcal{T}_\sigma$ denote the set of all standard Young tableaux of shape σ. Then, in order to prove (3.12), we will establish the bijection

$$(3.14) \qquad [3n-1] \times \mathcal{T}_{\mathcal{D}_{3n-2}} \to \mathcal{T}_{\mathcal{D}_{3n-1}} \dot{\cup} \mathcal{T}_{\mathcal{D}^*_{3n-1}}.$$

Given a pair (T,i) where $i \in [3n-1]$ and T is a standard Young tableau of shape $\mathcal{D}_{3n-2}$ with entries from the set $[3n-1] - \{i\}$. Suppose that the rightmost and topmost box α of T has entry q. If $i < q$, then we put a box with entry i on the top of box α, which gives us a standard Young tableau of shape $\mathcal{D}_{3n-1}$ with entries from 1 to $3n-1$. Otherwise we put a box with entry i to the right of box α, which, after transposing the rows into columns, is a standard Young tableau of shape $\mathcal{D}^*_{3n-1}$ with entries from 1 to $3n-1$. It is clear that this procedure is invertible, so the bijection (3.14) follows. Furthermore, it holds that $f^{\mathcal{D}_{3i-1}} = f^{\mathcal{D}^*_{3i-1}}$ since for any standard Young tableau of shape $\mathcal{D}_{3n-1}$, if we replace every entry q by $3n-q$ and flip the diagram $\mathcal{D}_{3n-1}$ upside-down and reverse it left-to-right, we obtain a standard Young tableau of shape $\mathcal{D}^*_{3n-1}$. In combination of (3.14), it follows that (3.12) is true.

In order to prove (3.13), we next establish the bijection

$$(3.15) \qquad [3n] \times \mathcal{T}_{\mathcal{D}^*_{3n-1}} \to \mathcal{T}_{\mathcal{D}_{3n}} \dot{\cup} \mathcal{T}_{\mathcal{C}_{3n}}$$

which is analogous to (3.14). Given a pair (T,i) where $i \in [3n]$ and T is a standard Young tableau of shape $\mathcal{D}^*_{3n-1}$ with entries from the set $[3n] - \{i\}$. Suppose that the rightmost and topmost box α of T has entry q. If $i < q$, then we put a box with entry i on the top of box α, which gives us a standard Young tableau of shape $\mathcal{D}_{3n}$ with entries from 1 to $3n$. Otherwise we put a box with entry i to the right of box α, which is a standard Young tableau of shape $\mathcal{C}_{3n}$ with entries from 1 to $3n$. This implies that (3.15) is a bijection, thus in view of $f^{\mathcal{D}_{3n-1}} = f^{\mathcal{D}^*_{3n-1}}$, (3.13) holds. $\square$

By Lemma 11 it suffices to count the numbers $f^{\mathcal{D}_{3n-2}}$ and $f^{\mathcal{C}_{3n}}$. Consider the boxes

$$(1,n-1),(2,n-2),\ldots,(n-1,1)$$

of the 3-strip diagram $\mathcal{D}_{3n-2}$, one of these boxes has the minimal entry 1 for any standard Young tableau from $\mathcal{T}_{\mathcal{D}_{3n-2}}$. Let $\mathcal{D}_{3n-2,i}$ be the 3-strip diagram $\mathcal{D}_{3n-2}$ after removing the box $(i, n-i)$. Then we have

Lemma 12. *For $1 \leq i \leq n-1$, the numbers $f^{\mathcal{D}_{3n-2,i}}$ satisfy*

$$(3.16) \qquad (3n-2)f^{\mathcal{D}_{3n-2,i}} = f^{\mathcal{D}_{3n-2}} + \binom{3n-2}{3i-1}f^{\mathcal{D}_{3i-1}}f^{\mathcal{D}_{3n-3i-1}}.$$

Proof. Let $\mathcal{S}$ denote the set of all $(3i-1)$-subsets of $[3n-2]$, we aim to construct the bijection

$$(3.17) \qquad [3n-2] \times \mathcal{T}_{\mathcal{D}_{3n-2,i}} \to \mathcal{T}_{\mathcal{D}_{3n-2}} \dot{\cup} (\mathcal{S} \times \mathcal{T}_{\mathcal{D}^*_{3i-1}} \times \mathcal{T}_{\mathcal{D}^*_{3n-3i-1}}),$$

from which (3.16) follows immediately. Given a pair (T, r) where $r \in [3n-2]$ and T is a standard Young tableau of shape $\mathcal{D}_{3n-2,i}$ with entries from the set $[3n-2] - \{r\}$. Suppose that the entries of box $(i+1, n-i)$ and box $(i, n-i+1)$ are q_1 and q_2 in T, we set $q = \min\{q_1, q_2\}$. If $r < q$, then we add a box $(i, n-i)$ with entry r to T, which is a standard Young tableau of shape $\mathcal{D}_{3n-2}$.

If $r > q = q_1$, then we consider a segment of T from the starting box of $\mathcal{D}_{3n-2,i}$ to box $(i+1, n-i)$ and we add a box with entry r to the right of box $(i+1, n-i)$, which, after transposing the rows into columns, leads to a standard Young tableau of shape $\mathcal{D}^*_{3n-3i-1}$ with entries coming from a $(3n-3i-1)$-subset A of $[3n-2]$. Moreover, the segment of T from box $(i+1, n-i+1)$ to the ending box of $\mathcal{D}_{3n-2,i}$, is a standard Young tableau of shape $\mathcal{D}^*_{3i-1}$ with entries coming from the complement set A^c of A with respect to $[3n-2]$.

If $r > q = q_2$, then we consider a segment of T from box $(i, n-i+1)$ to the ending box of $\mathcal{D}_{3n-2,i}$ and we add a box with entry r right below the box $(i, n-i+1)$, which leads to a standard Young tableau of shape $\mathcal{D}^*_{3i-1}$ with entries coming from a $(3i-1)$-subset B of $[3n-2]$. Moreover, the segment of T from the starting box of $\mathcal{D}_{3n-2,i}$ to box $(i+1, n-i+1)$, which, after transposing the rows into columns, is a standard Young tableau of shape $\mathcal{D}^*_{3n-3i-1}$ with entries coming from the complement set B^c of B with respect to $[3n-2]$.

Conversely, given a standard Young tableau T_0 of shape $\mathcal{D}_{3n-2}$, we set r to be the entry of box $(i, n-i)$ in T_0 and after we remove box $(i, n-i)$ from T_0, we obtain a standard Young tableau of shape $\mathcal{D}_{3n-2,i}$. Given a triple (D, T_1, T_2) where T_1 is a standard Young tableau of shape $\mathcal{D}^*_{3i-1}$ with entries from $D \in \mathcal{S}$, and T_2 is a standard Young tableau of shape $\mathcal{D}^*_{3n-3i-1}$ with entries from the complement set D^c.

Suppose that the entry of box $(i, 1)$ in T_1 is q_3 and the entry of box $(n-i, 1)$ in T_2 is q_4, if $q_3 > q_4$, we remove the box $(n-i+1, 1)$ of T_2, then transpose it from columns into rows and put the box with entry q_4 to the left of box $(i+1, 1)$ of T_1. This gives us a standard Young tableau of shape $\mathcal{D}_{3n-2,i}$ such that the entry of box $(i, n-i+1)$ is larger than the one of box $(i+1, n-i)$ and we choose r to be the entry of box $(n-i+1, 1)$ of T_2, so that $r > q_4$.

If $q_3 < q_4$, we transpose T_2 from columns into rows, then put its rightmost and topmost box right below the box with entry q_3 after we remove the box $(i+1, 1)$ of T_1. This gives us a standard Young tableau of shape $\mathcal{D}_{3n-2,i}$ such that the entry of box $(i, n-i+1)$ is

smaller than the one of box $(i+1, n-i)$ and we choose r to be the entry of box $(i+1,1)$ of T_1, so that $r > q_3$.

Since all cases are disjoint and cover all possible scenarios, (3.17) is a bijection. In view of $f^{\mathcal{D}_{3i-1}} = f^{\mathcal{D}^*_{3i-1}}$, (3.16) follows. $\qquad\square$

Example 17. *For $i = 2$, we consider the pairs $(T_1, 4)$ and $(T_2, 6)$. Since $4 < \min\{6, 8\}$, we put a box with entry 4 to T_1. Since $6 > \min\{4, 8\}$, we separate T_2 of shape $\mathcal{D}_{13,2}$ into two standard Young tableaux of shapes D^*_5 and D^*_8.*
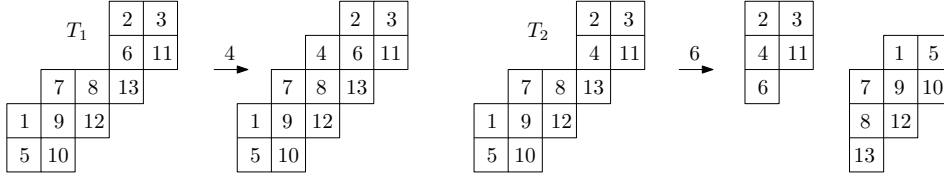
FIGURE 3.5. Two examples of the bijection (3.17) in Lemma 12.

Let $\mathcal{C}_{3n,i}$ be the 3-strip diagram $\mathcal{C}_{3n}$ after removing the box $(i, n-i)$, we can decompose the skew diagram $\mathcal{C}_{3n}$ in exactly the same way. So we omit the proof of Lemma 13.

Lemma 13. *For $1 \leq i \leq n-1$, the numbers $f^{\mathcal{C}_{3n,i}}$ satisfy*

$$(3.18) \qquad (3n)f^{\mathcal{C}_{3n,i}} = f^{\mathcal{C}_{3n}} + \binom{3n}{3i}f^{\mathcal{C}_{3i}}f^{\mathcal{C}_{3n-3i}}.$$

With the help of recursions (3.16) and (3.18), we could use the generating function approach to finally derive the numbers of 3-strip tableaux.

Proof of (3.6)-(3.8). Summing (3.16) and (3.18) over all i gives us

$$(3.19) \qquad (2n-1)f^{\mathcal{D}_{3n-2}} = \sum_{i=1}^{n-1}\binom{3n-2}{3i-1}f^{\mathcal{D}_{3i-1}}f^{\mathcal{D}_{3n-3i-1}}$$

$$(3.20) \qquad (2n+1)f^{\mathcal{C}_{3n}} = \sum_{i=1}^{n-1}\binom{3n}{3i}f^{\mathcal{C}_{3i}}f^{\mathcal{C}_{3n-3i}}.$$

We can translate the recursions (3.19) and (3.20) into two identities of exponential generating functions. We define that

$$f(x) = \sum_{n\geq 1}\frac{f^{\mathcal{D}_{3n-2}}}{(3n-2)!}x^{2n-1}, \ \ g(x) = \sum_{n\geq 1}\frac{f^{\mathcal{D}_{3n-1}}}{(3n-1)!}x^{2n-1}, \ \ h(x) = \sum_{n\geq 1}\frac{f^{\mathcal{C}_{3n}}}{(3n)!}x^{2n-1}.$$

From (3.12) we have $f(x) = 2g(x)$. Furthermore, (3.19) is equivalent to $f'(x) = 1 + g(x)^2$ where $f(0) = 0$. This leads to a unique solution, $g(x) = \tan(x/2)$. Together with the exponential generating function for A_{2n-1}; see (3.1), we can prove (3.6) and (3.7). Similarly,

(3.20) is equivalent to $-h'(x) = -1 + \frac{2}{x}h(x) - h^2(x)$ where $h(0) = 1$. This yields a unique solution

$$h(x) = -\frac{1}{\tan x} + \frac{1}{x} = \frac{1}{3}x + \frac{1}{45}x^3 + \cdots,$$

from which we can derive the numbers $f^{\mathcal{C}_{3n}}$ by expanding $h(x)$, i.e.,

(3.21) $$f^{\mathcal{C}_{3n}} = \frac{(3n)!A_{2n-1}}{(2n-1)!(2^{2n}-1)} = (3n)!\tilde{A}_{2n-1}.$$

Together with (3.13) and (3.7), (3.8) is proved.

3.2.4. *Proof of (3.9)-(3.10).* For the 4-strip diagram $\mathcal{D}_4(\varnothing; \varnothing)$, we choose another outside decomposition $\phi^* = (\theta_1^*, \theta_2^*, \ldots, \theta_k^*)$, which is slightly different to the one $\phi = (\theta_1, \theta_2, \ldots, \theta_k)$ for the $2k$-strip diagram $\mathcal{D}_{2k}(\tilde{\lambda}; \tilde{\mu})$. The benefit to make such a slight change is that the determinant in (3.4) is further simplified, which only has the numbers $\bar{A}_i$ as entries.

We call a 2-strip *a zig-zag strip* if the corresponding standard Young tableaux are in bijection with up-down permutations. For instance, the 2-strip diagram $\mathcal{D}_2(\varnothing; \varnothing)$ is a zig-zag strip and all strips in Fig. 3.6 are zig-zag strips. For the 4-strip diagrams $\mathcal{D}_4(\varnothing; \varnothing)$,
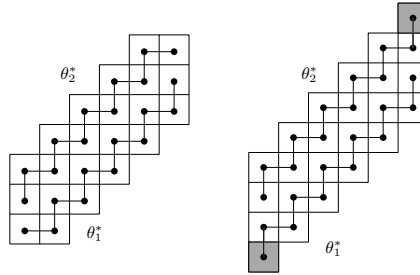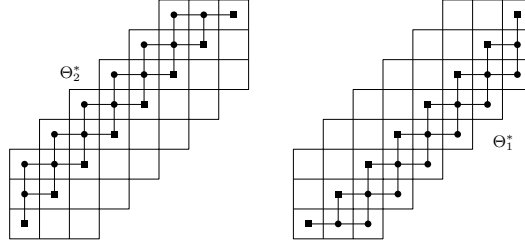

FIGURE 3.6. The outside nested decompositions $\phi^* = (\theta_1^*, \theta_2^*)$ of the 4-strip diagrams $\mathcal{D}_4(\varnothing; \varnothing)$ (left) and $\mathcal{D}_4((1); (1))$ (right).

we can peel each diagram off into successive maximal zig-zag outer strips $\theta_1^*, \theta_2^*, \ldots, \theta_k^*$ beginning from the outside. See the left one in Fig. 3.6. It is clear that the numbers of any zig-zag strip θ_1^* or θ_2^* are Euler numbers or tangent numbers. By Definition 1.9 we find that

$$f^{\theta_1^*} = f^{\theta_2^*} = A_{2n-1}, \quad f^{\theta_1^* \# \theta_2^*} = A_{2n-2} \quad \text{and} \quad f^{\theta_2^* \# \theta_1^*} = A_{2n}.$$

By Corollary 3, we can prove (3.9) and (3.10) follows in the same way.

3.2.5. *Proof of (3.11).* For the 5-strip diagram $\mathcal{D}_5(\varnothing; \varnothing)$, we choose another outside nested decomposition $\Phi^* = (\Theta_1^*, \Theta_2^*, \ldots, \Theta_k^*)$, which is slightly different to the one $\Phi = (\Theta_1, \Theta_2, \ldots, \Theta_k)$ for the $(2k+1)$-strip diagram $\mathcal{D}_{2k+1}(\tilde{\lambda}; \tilde{\mu})$. The benefit to make such a slight change is that the determinant in (3.5) is further simplified, which only has the numbers $\hat{A}_i$ and $\tilde{A}_j$ as entries.

We call a 3-strip *a zig-zag thickened strip* if the number of such 3-strip tableaux is one of the numbers $f^{\mathcal{D}_{3n-2}}$, $f^{\mathcal{D}_{3n-1}}$, $f^{\mathcal{D}_{3n}}$ and $f^{\mathcal{C}_{3n}}$. For instance, two thickened strips in Fig. 3.7

are zig-zag thickened strips. It is clear that the numbers of the zig-zag thickened strips Θ_1^* or Θ_2^* are $f^{\mathcal{C}_{3n-3}}$. By Definition 1.9 we find that

$$f^{\Theta_1^*} = f^{\Theta_2^*} = f^{\mathcal{C}_{3n-3}} \quad \text{and} \quad f^{\Theta_1^* \# \Theta_2^*} = f^{\Theta_2^* \# \Theta_1^*} = f^{\mathcal{D}_{3n-3}} = f^{\mathcal{D}_3((1);(1))}.$$

By Corollary 3, we can prove that

$$f^{\mathcal{D}_5(\varnothing;\varnothing)} = \frac{(5n-6)!}{(3n-3)!^2}((f^{\mathcal{C}_{3n-3}})^2 - (f^{\mathcal{D}_3((1);(1))})^2).$$

Combining (3.8) and (3.21), we can conclude that (3.11) is true.

FIGURE 3.7. The outside nested decomposition $\Phi^* = (\Theta_1^*, \Theta_2^*)$ of the 5-strip diagram $\mathcal{D}_5(\varnothing;\varnothing)$.

ACKNOWLEDGEMENT

4. APPENDIX

4.1. **Proof of Lemma 6.**

Proof. We shall prove the equivalent statement, namely, if $\pi \in S_g$ and $\pi \neq$ id, then any g-tuple (2.1) of double lattice path is crossing.

First we consider a total order $\prec$ of all starting points $u_1, u_2, \ldots, u_g$ and a total order $\prec$ of all ending points $v_1, v_2, \ldots, v_g$ of the double lattice paths. For every i, let $x(u_i)$ and $y(u_i)$ denote the x-th coordinate and y-th coordinate of point u_i, similarly for $x(v_i)$ and $y(v_i)$. We recall that $y(u_i), y(v_i) \in \{1, \infty\}$ according to Definition 2.1. We define $u_s \prec u_i$ if and only if one of the following conditions is true:

(1) $\infty = y(u_s) > y(u_i) = 1$;

(2) $y(u_s) = y(u_i) = \infty$ and $x(u_s) > x(u_i)$;

(3) $y(u_s) = y(u_i) = 1$ and $x(u_s) < x(u_i)$.

We define $v_s \prec v_i$ if and only if one of the following conditions is true:

(4) $\infty = y(v_s) > y(v_i) = 1$;

(5) $y(v_s) = y(v_i) = \infty$ and $x(v_s) < x(v_i)$;

(6) $y(v_s) = y(v_i) = 1$ and $x(v_s) > x(v_i)$.

We claim that for any i and s, $u_s \prec u_i$ holds if and only if $v_s \prec v_i$ holds. The essential reason for this is the fact that Φ is an outside thickened strip decomposition (Definition 1.5), so when we read the boxes on the bottom perimeter and the left perimeter of the skew shape λ/μ in the right-to-left and bottom-to-top order, the starting box of Θ_s comes earlier than the starting box of Θ_i if and only if $u_s \prec u_i$ holds. Since one thickened strip is on the right side or the bottom side of the other thickened strip; see Definition 1.5, when we read the boxes on the right perimeter and the top perimeter of the skew shape λ/μ in the bottom-to-top and right-to left order, the ending box of Θ_s comes earlier than the ending box of Θ_i if and only if $v_s \prec v_i$ holds. This implies that for any i and s, $u_s \prec u_i$ holds if and only if $v_s \prec v_i$ holds.

Second, for any π such that id $\neq \pi \in S_g$, there exist two integers s and t such that $u_{\pi_s} \prec u_{\pi_t}$ and $v_t \prec v_s$ because otherwise it contradicts the assumption $\pi \neq$ id. We wish to show that $P(u_{\pi_s}, v_s)$ and $P(u_{\pi_t}, v_t)$ are crossing, which can be proved by discussing all cases when one of the previous conditions (1)-(3) for $u_{\pi_s} \prec u_{\pi_t}$ is true, and one of the previous conditions (4)-(6) for $u_t \prec u_s$ is true. So we conclude that if $\pi \in S_g$ and $\pi \neq$ id, then (2.1) is crossing. $\qquad \square$

References

[1] D. ANDRÉ, Développement de sec x et tan x, Comptes rendus de l'Académie des sciences, 88, 965-979, 1879.

[2] Y. BARYSHNIKOV AND D. ROMIK, Enumeration formulas for Young tableaux in a diagonal strip, Israel Journal of Mathematics 178, 157-186, 2010.

[3] W.Y.C. CHEN, G.G YAN AND A.L.B YANG, Transformations of border strips and Schur function determinants, J. Algebr. Comb. 21, 379-394, 2005.

[4] N. ELKIES, On the sums $\sum_{k=-\infty}^{\infty} (4k+1)^{-n}$, American Mathematical Monthly, 110, 561-573, 2003.

[5] G.Z. GIAMBELLI, Alcune proprietà dele funzioni simmetriche caratteristiche, Atti Torino, 38, 823-844, 1903.

[6] I.M. GESSEL AND X.G. VIENNOT, Determinants, Paths and Plane Partitions, preprint, 1989.

[7] A.M. HAMEL, Determinant forms for symplectic and orthogonal Schur functions, Canad. J. Math., 49, 263-282, 1997.

[8] A.M. HAMEL AND I.P. GOULDEN, Planar decompositions of tableaux and Schur function determinants, Europ. J. Combinatorics, 16, 461-477, 1995.

[9] C. JACOBI, De functionibus alternantibus earumque divisione per production e differentiis elementorum conflatum, J. Reine Angew. Math., 22, 360-371, 1841. Also in Mathematische Werke, Vol. 3, Chelsea, New York, 439-452, 1969.

[10] A. LASCOUX AND P. PRAGACZ, Ribbon Schur Functions, Europ. J. Combinatorics, 9(6), 561-574, 1988.

[11] I.G. MACDONALD, Symmetric functions and Hall polynomials, Oxford University Press, 1979.

[12] A.H. Morales, I. Pak and G. Panova, Hook formulas for skew shapes, 2015, arXiv:1512.08348v1.

[13] R. Stanley, Enumerative Combinatorics I and II, Cambridge University Press, 1999.

[14] R. Stanley, The rank and minimal border strip decompositions of a skew partition, J. Combin. Theory Ser. A, 100, 349-375, 2002.

[15] J.R. Stembridge, Nonintersecting paths, pfaffians and plane partitions, Adv. Math., 83, 96-131, 1990.

[16] K. Ueno, Lattice path proof of the ribbon determinant formula for Schur functions, Nagoya Math. J., 124, 55-59, 1991.

[17] G.G. Yan, A.L.B. Yang and J. Zhou, The Zrank conjecture and restricted Cauchy matrices, Linear Algebra Appl. 411, 371-385, 2005.

Institut für Diskrete Mathematik und Geometrie, TU Wien, Wiedner Hauptstr. 8–10, 1040 Vienna, Austria