

# ASYMPTOTICS AND STATISTICS ON FISHBURN MATRICES: DIMENSION DISTRIBUTION AND A CONJECTURE OF STOIMENOW

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**ABSTRACT.** We establish the asymptotic normality of the dimension of large-size random Fishburn matrices by a complex-analytic approach. The corresponding dual problem of size distribution under large dimension is also addressed and follows a quadratic type normal limit law. These results represent the first of their kind and solve two open questions raised in the combinatorial literature. They are presented in a general framework where the entries of the Fishburn matrices are not limited to binary or nonnegative integers. The analytic saddle-point approach we apply, based on a powerful transformation for  $q$ -series due to Andrews and Jelínek, is also useful in solving a conjecture of Stoimenow in Vassiliev invariants.

*Keywords:* Fishburn matrices; dimension; saddle-point method;  $q$ -series; a conjecture of Stoimenow.

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## 1. INTRODUCTION AND MAIN RESULTS

Fishburn matrices (abbreviated as FMs), introduced by Peter Fishburn in 1970 during his study of interval orders [9], are upper-triangular square matrices with nonnegative integers as entries such that no row and no column contains exclusively zeros. They also appeared a few years later under a different guise in the study of transitively directed graphs by Andresen and Kjeldsen [1], where essentially a recursive formula was given on the number of *primitive FMs* (FMs with entries 0 or 1) with respect to the dimension and the first row sum (which is  $\xi(n, k)$  in [1]; see also § 5.2). For example, all FMs with *size* (or sum of all entries) equal to 4 are depicted in Figure 1.1 and all primitive FMs of dimension 3 in Figure 1.2.

$$\begin{array}{c}
 \binom{4}{1} \binom{12}{1} \binom{21}{1} \binom{11}{2} \binom{20}{2} \binom{30}{1} \binom{10}{3} \\
 \binom{110}{10} \binom{101}{10} \binom{100}{11} \binom{200}{10} \binom{100}{20} \binom{100}{10} \binom{110}{01} \binom{1000}{100} \\
 \binom{1}{1} \binom{1}{1} \binom{1}{1} \binom{1}{1} \binom{1}{1} \binom{2}{2} \binom{1}{1} \binom{10}{1}
 \end{array}$$

**Figure 1.1.** All 15 FMs of size  $n = 4$ . The average dimension of these matrices is  $\frac{1}{15}(1 \cdot 1 + 2 \cdot 6 + 3 \cdot 7 + 4 \cdot 1) \approx 2.533$ , which is already close to our asymptotic approximation  $\frac{6}{\pi^2}n = \frac{24}{\pi^2} \approx 2.432$  in Theorem 1.

$$\binom{100}{10} \binom{110}{10} \binom{100}{11} \binom{101}{10} \binom{110}{11} \binom{111}{10} \binom{101}{11} \binom{111}{11} \binom{110}{01} \binom{111}{01} \\
 \binom{1}{1} \binom{1}{1} \binom{1}{1} \binom{1}{1} \binom{1}{1} \binom{1}{1} \binom{1}{1} \binom{1}{1} \binom{1}{1} \binom{1}{1}$$

**Figure 1.2.** All 10 primitive FMs of dimension  $n = 3$ . The average size (sum of entries) of these matrices is  $\frac{1}{10}(3 \cdot 1 + 4 \cdot 4 + 5 \cdot 4 + 6 \cdot 1) = 4.5$  while the asymptotic average size equals  $\frac{1}{4}n(n+1) = 3$  in Theorem 3.


Apart from the connection between primitive FMs and transitively directed graphs, it is now known that FMs are in bijection with interval orders,  $(\mathbf{2+2})$ -free posets, ascent sequences, certain pattern-avoiding permutations and regular linearized chord diagrams (regular LCDs), etc.; see for instance [4, 8, 10].

The numbers of FMs of a given size are known as the *Fishburn numbers* (see [7] and [20, A022493]), which can be computed by the Taylor coefficients of the generating function

$$(1.1) \quad \sum_{k \geq 0} \prod_{1 \leq j \leq k} (1 - (1 - z)^j) = 1 + z + 2z^2 + 5z^3 + 15z^4 + 53z^5 + 217z^6 + \dots$$

This (formal) generating function was derived by Zagier [23], using a recursive formula found earlier by Stoimenow [22] for the number of regular LCDs with a given length; we postpone the exact definition of LCDs and regular LCDs to Section 4. Stoimenow also made in the same paper [22] a conjecture concerning the asymptotic relation between the Fishburn numbers and the number of connected regular LCDs of size  $n$ , which will be addressed in more detail at the end of this section.

Since the seminal work [4] by Bousquet-Mélou, Claesson, Dukes and Kitaev, much attention has been drawn to the refined enumeration of Fishburn structures with respect to various classical statistics; see for instance [4, 8, 12, 15, 16, 17, 18, 19]. Two types of statistics among all members of the Fishburn family are Eulerian and Stirling statistics [12]: any statistic whose distribution over a member of the Fishburn family equals the distribution of the dimension (resp. the first row sum) on FMs is called an *Eulerian* (resp. a *Stirling*) statistic; see Table 1 for a summary of the equidistributed Eulerian and Stirling statistics on six Fishburn structures.

Fishburn structures	Eulerian statistics	Stirling statistics
FMs	dimension – 1	sum of the first row (or the last column) number of weakly northeast cells
(2+2)-free posets	magnitude – 1	number of minimal elements
Ascent sequences	asc, rep	zero, max, rmin
(2–1)-avoiding sequences	rep	max
(  )-avoiding permutations	des, iasc	lmin, lmax, rmax
Regular linearized chord diagrams	length of the initial run of openers	number of pairs of arcs ( $a, b$ ), ( $c, d$ ) such that $a < b = c - 1 < d - 1$

**Table 1.** *Equidistributed Eulerian and Stirling Statistics on Fishburn structures: statistics in the second (resp. third) and column are all equidistributed with each other; see [4, 8, 12, 15, 19] for precise definitions.*

While there is a large literature on the combinatorial aspects of statistics over Fishburn structures, very few studies have been conducted on asymptotic and stochastic properties concerning structures of large size; see [6, 23] and our previous paper [14]. Questions such as (see [15]) “*what is the expected dimension of a random FM of size  $n$  when each of the size- $n$  FMs is chosen with the same probability?*” and “*what is the expected size of a random FM when all FMs of the same dimension are equally likely?*” have remained open, and the primary purpose of this paper is to answer these questions in a more *complete* (including the variance and the limiting distribution) and more *systematic* (covering a wide class of generalized FMs) way.

In contrast to the Stirling statistics worked out in [14], which have typically *logarithmic behaviors* (logarithmic mean and logarithmic variance), the Eulerian statistics studied in this paper, namely,

dimension distribution with fixed size, have asymptotically *linear mean and linear variance* (the corresponding dual statistic, size distribution of fixed dimension, is *quadratic*). Such a contrast is well known for statistics on permutations, but has remained mostly elusive on Fishburn structures. Whichever the case, the limiting distribution of any statistic is normal as long as the variance goes unbounded, although the proof technicalities differ.

Since an FM of a given size can be viewed as an integer partition (but allowing 0 as entries) arranged on an upper-triangular matrix, there is yet a third class of Poisson statistics examined in detail in [14]: the number of occurrences of the smallest nonzero entry in the matrix. Similar to the classical integer partitions where 1 has a predominant frequency, the smallest nonzero entry in FMs also appears almost everywhere. But different from the exponential limit law of the occurrences of the smallest part in random integer partitions, the smallest entry in FMs has its occurrences following mostly (but not always) a Poisson limit law; see [14]. This indicates an even higher concentration of the smallest entry near its expected value in the context of random FMs. Such a viewpoint will also be useful in interpreting our asymptotic results in this paper.

The approach developed in [14] relies on a direct two-stage saddle-point method that is applied to the generating functions with a sum-of-product form, and is very powerful in that it is not only applicable to the asymptotics of a wide class of concrete examples, but also provides an effective means of understanding the limit laws of Stirling statistics. In the present paper, we further extend the same saddle-point approach to Eulerian statistics. This extension is however not straightforward as a direct application fails due to the violent fluctuations in summing the dominant terms, similar to the summands on the left-hand side of (1.1). It turns out that a key property needed is a generalized Rogers-Fine identity derived by Andrews and Jelínek in [2]. Furthermore, an additional difficulty arises in handling the uniformity in the extra parameter of the probability generating function.

Given any multiset  $\Lambda$  of nonnegative integers with the generating function

$$(1.2) \quad \Lambda(z) = 1 + \lambda_1 z + \lambda_2 z^2 + \dots,$$

a  $\Lambda$ -FM is an FM with entry set  $\Lambda$ . The original FMs correspond to the situation when all  $\lambda_j$ 's equal 1, and the primitive FMs to  $\lambda_j = \delta_{j,1}$ ,  $j \geq 1$ , the Kronecker symbol. Although such a matrix formulation requires that all the coefficients  $\lambda_j$  be nonnegative integers, our proof is independent of this restriction and the  $\lambda_j$ 's can indeed be any nonnegative reals.

It is known that if  $\Lambda(z)$  is analytic at  $z = 0$  with  $\lambda_1 > 0$  then the number of  $\Lambda$ -FMs of size  $n$  is given by (see [14])

$$(1.3) \quad a_n := [z^n] \sum_{k \geq 0} \prod_{1 \leq j \leq k} (1 - \Lambda(z)^{-j}) = cn^{\frac{1}{2}} (\lambda_1 \mu)^n n! (1 + O(n^{-1})),$$

where

$$(1.4) \quad (c, \mu) := \left( \frac{12\sqrt{3}}{\pi^{5/2}} e^{\frac{\pi^2}{6} \left( \frac{\lambda_2}{\lambda_1^2} - \frac{1}{2} \right)}, \frac{6}{\pi^2} \right).$$

Here  $[z^n]f(z)$  denotes the Taylor coefficient of  $f(z)$ . We see that the dominant asymptotic order (neglecting the leading constant  $c$ ) depends crucially on  $\lambda_1$ , but not on any other  $\lambda_j$ 's with  $j \geq 2$ , showing roughly the pervasiveness of 1 in a typical  $\Lambda$ -FM. On the other hand, the expression of  $c$ , as well as the violent cancellations of terms when summing the Taylor expansions of the finite products on the left-hand side of (1.3), implicitly points to the difficulty of the analysis involved; see [14] for more precise results.

1.1. **Dimension distribution of fixed-size FMs.** Define the bivariate generating function (see [12, 15])

$$(1.5) \quad F(z, v) := \sum_{n \geq 0} P_n(v) z^n = \sum_{k \geq 0} \prod_{1 \leq j \leq k} \left( 1 - \frac{1}{1 + v(\Lambda(z)^j - 1)} \right),$$

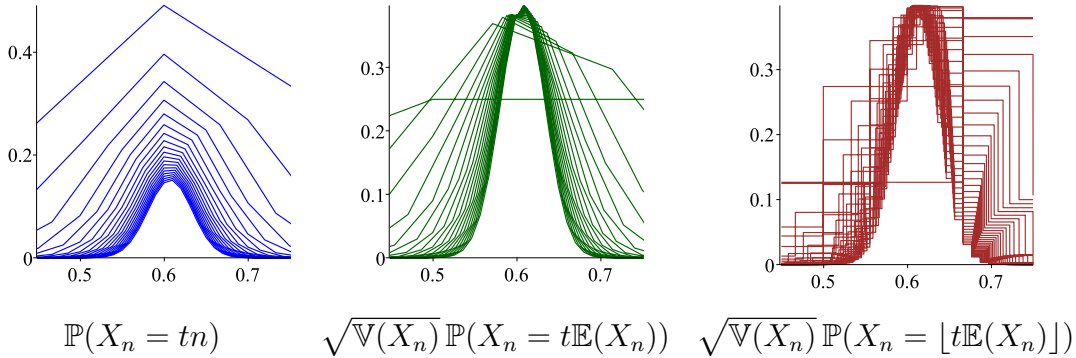
as an extension of the generating function in (1.3), where  $P_n(v)$  is the generating polynomial of the dimension of size- $n$   $\Lambda$ -FMs with  $P_n(1) = a_n$ .

**Theorem 1** (Open problem of [6]). *Assume that  $\Lambda(z)$  is analytic at  $z = 0$  with  $\lambda_1 > 0$  and that all  $\Lambda$ -FMs of size  $n$  are equally likely to be selected. Then the dimension  $X_n$  of a random matrix is asymptotically normally distributed with mean and variance both linear in  $n$ , namely,*

$$(1.6) \quad \frac{X_n - \mu n}{\sigma \sqrt{n}} \xrightarrow{d} \mathcal{N}(0, 1), \quad \text{with } (\mu, \sigma^2) := \left( \frac{6}{\pi^2}, \frac{3(12 - \pi^2)}{\pi^4} \right),$$

where the symbol  $\xrightarrow{d}$  stands for convergence in distribution and  $\mathcal{N}(0, 1)$  the standard normal distribution.

See Figure 1.3 for three different graphic renderings of the histograms of  $X_n$  when  $\Lambda = \mathbb{N}$ . Note that  $\sigma^2 = \mu^2 - \frac{1}{2}\mu$ , and the coefficient pair  $(\mu, \sigma^2)$  is to some extent universal as we will also see its occurrences in other classes of FMs (albeit in slightly different scales).



**Figure 1.3.** Three different ways of visualizing the asymptotic normality of  $X_n$  where we plot the histograms of  $X_n$  in the case when  $\Lambda(z) = (1 - z)^{-1}$ : for  $n = 5j$ ,  $1 \leq j \leq 20$  (left and middle) and  $n = 3k$ ,  $1 \leq k \leq 33$  (right).

What is particularly remarkable here is that the central limit theorem (1.6) is *independent* of  $\Lambda$  (as long as  $\lambda_1 > 0$ ). The same also holds true for the first row sum (see [14]), which behaves asymptotically like a normal distribution with both mean and variance asymptotic to  $\log n$ . Such an “invariance property” (1.6) may seem more surprising than its logarithmic counterpart because linear statistics cover stochastically a wider range of variations. We can view this phenomenon from a few different angles.

First, from the asymptotic approximation (1.3), we see that the number of general  $\Lambda$ -FMs with  $\lambda_1 > 0$  behaves roughly (modulo the leading constant  $c$ ) like  $\lambda_1^n$  times the number of primitive FMs of the same size with  $\Lambda(z) = 1 + z$ . So we next examine more closely how the magic constant  $\mu$

appears in random primitive FMs. The number of primitive FMs of size  $n$  is given by (see [20, A138265])

$$(a_n)_{n \geq 1} = (1, 1, 2, 5, 16, 61, 271, 1372, 7795, 49093, 339386, 2554596, \dots),$$

and it turns out that in this special case, we have an unexpected *identity* for the expected dimension:

$$(1.7) \quad \mu_n := \mathbb{E}(X_n) = \frac{a_{n+1}}{a_n} \quad (n \geq 1);$$

see Section 2 for a more general form as well as a combinatorial proof of (1.7); in other words, *the sum of the dimensions of all size- $n$  primitive FMs matrices equals the number of size- $(n + 1)$  primitive FMs*. In view of (1.3) and (1.7), we immediately get the asymptotic linearity of  $\mathbb{E}(X_n)$  with the mean constant  $\mu$ . In a similar manner, the second moment (and then the variance  $\sigma^2$ ) can be approached via the same analytic and combinatorial arguments:

$$\sum_{1 \leq k \leq n} \binom{k+1}{2} p_{n,k} + \sum_{1 \leq k \leq n+1} k^2 p_{n+1,k} = a_{n+3},$$

where  $p_{n,k}$  denotes the number of primitive FMs of size  $n$  and dimension  $k$ , which is  $[v^k]P_n(v)$  from (1.5) when  $\Lambda(z) = 1 + z$ , appearing also in [20, A137252].

In addition, we will also derive finer asymptotic approximations for  $\mathbb{E}(X_n)$  and  $\mathbb{V}(X_n)$ .

**Theorem 2.** *The mean and the variance of the dimension  $X_n$  (defined in Theorem 1) of a random  $\Lambda$ -FM of size  $n$  satisfy*

$$(1.8) \quad \mathbb{E}(X_n) = \mu \left( n + \frac{3}{2} \right) - \frac{\lambda_2}{\lambda_1^2} + O(n^{-1}),$$

$$(1.9) \quad \mathbb{V}(X_n) = \sigma^2 \left( n + \frac{3}{2} \right) - \frac{1}{4} + \frac{\lambda_2}{2\lambda_1^2} + O(n^{-1}),$$

where  $(\mu, \sigma^2)$  is given in (1.6).

Note that the dependence of  $\mathbb{E}(X_n)$  and  $\mathbb{V}(X_n)$  on  $\Lambda$  is weak: only the ratio of  $\lambda_2$  and  $\lambda_1^2$  appears in the constant terms, and similarly for higher central moments. For example,

$$\mathbb{E}(X_n - \mu_n)^3 = \frac{\pi^4 - 54\pi^2 + 432}{\pi^6} \left( n + \frac{3}{2} \right) + \frac{1}{12} - \frac{\lambda_2}{6\lambda_1^2} + O(n^{-1}),$$

$$\mathbb{E}(X_n - \mu_n)^4 = 3\mathbb{V}(X_n)^2 + \left( 6\sigma^4 - \frac{\mu^2}{12} \right) n + O(1).$$

In principle, such calculations can be carried out further for all higher central moments and lead possibly to an alternative proof of (1.6) by the method of moments. But the cancellations involved in such a process are very heavy and complex, so we will instead work out an analytic, cancellation-free approach. Other  $\lambda_j$ 's will appear in lower-order terms.

Interestingly, the source of the seemingly strange but omnipresent ratio “ $\frac{\lambda_2}{\lambda_1^2}$ ” in the second-order terms will be indicated in Section 2.3.4.

The same types of normal limit results are expected to hold for other classes of FMs, and we will briefly examine two of them: self-dual  $\Lambda$ -FMs (or persymmetric, namely, symmetric with respect to the anti-diagonal), and  $\Lambda$ -FMs whose smallest nonzero entries are 2. The corresponding central limit theorems are summarized in the Table 2; see Section 6 for more information.

$\Lambda$ -FMs with $\lambda_1 > 0$ (Theorem 1)	Self-dual $\Lambda$ -FMs with $\lambda_1 > 0$ (Theorem 16)	$\Lambda$ -FMs with $\lambda_1 = 0, \lambda_2 > 0$ (Theorem 15)
$\mathcal{N}(\mu n, \sigma^2 n)$	$\mathcal{N}(\mu n, 2\sigma^2 n)$	$\mathcal{N}(\frac{1}{2}\mu n, \frac{1}{2}\sigma^2 n)$

**Table 2.** A summary of the central limit theorems for the dimension of different types of random  $\Lambda$ -FMs. Note specially the change in the mean and the variance coefficients: while the halving in the last column is well expected, the asymptotic doubling of the variance in the self-dual FMs comes as a little surprise.

**1.2. Size distribution of fixed-dimension FMs.** We now address a dual problem: the size distribution of random  $\Lambda$ -FMs with the same dimension. The problem is well-defined when  $\Lambda$  is finite and all coefficients of  $\Lambda(z)$  are positive integers.

**Theorem 3** (Extended open problem 5.5 of [15]). *Assume that  $\Lambda(z)$  is a polynomial with positive coefficients and  $\Lambda(1) \neq 1$ , and that all  $\Lambda$ -FMs of dimension  $m$  are equally likely. Then the size  $Y_m$  of a random matrix is asymptotically normally distributed with mean and variance both of order  $\Theta(m^2)$ :*

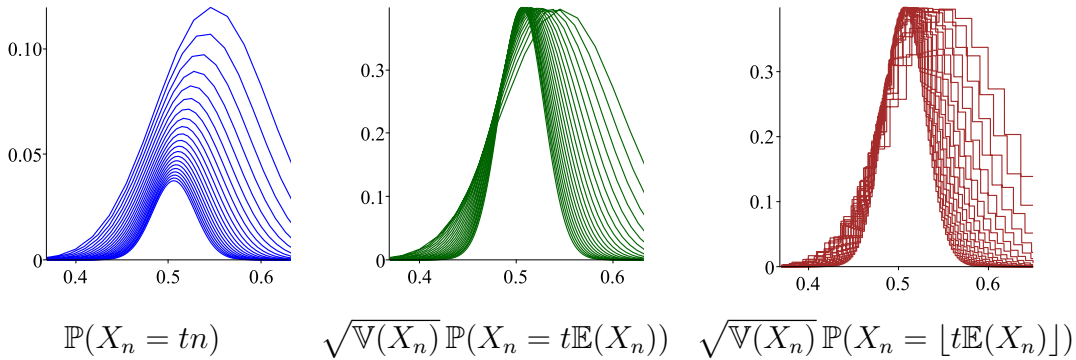
$$(1.10) \quad \frac{Y_m - \hat{\mu}m^2}{\hat{\sigma}m} \xrightarrow{d} \mathcal{N}(0, 1),$$

where  $\hat{\mu}, \hat{\sigma}^2 > 0$  are given by

$$(1.11) \quad (\hat{\mu}, \hat{\sigma}^2) := \left( \frac{\Lambda'(1)}{2\Lambda(1)}, \frac{1}{2} \left( \frac{\Lambda'(1) + \Lambda''(1)}{\Lambda(1)} - \left( \frac{\Lambda'(1)}{\Lambda(1)} \right)^2 \right) \right).$$

See Figure 1.4 for three different plots of the histograms of  $Y_m$  in the case of binary FMs for which  $(\hat{\mu}, \hat{\sigma}^2) = (\frac{1}{4}, \frac{1}{8})$ . Note that, if  $\Lambda(z) = 1 + \sum_{1 \leq j \leq \ell} \lambda_j z^j$  with  $\ell \geq 1$  is a positive polynomial, then

$$2\hat{\sigma}^2 = \frac{1}{\Lambda(1)} \sum_{1 \leq j \leq \ell} \left( j - \frac{\Lambda'(1)}{\Lambda(1)} \right)^2 \lambda_j > 0.$$



**Figure 1.4.** Three different ways of visualizing the asymptotic normality of  $Y_m$  where the histograms of  $Y_m$  are given in the case when  $\Lambda(z) = 1 + z$ : for  $10 \leq n \leq 30$ .

**Remark 1.** Define the random variable  $Y$  by  $\mathbb{E}(z^Y) = \frac{\Lambda(z)}{\Lambda(1)}$ . The quadratic behavior of  $Y_m$  naturally suggests the question: “what is the probability that a randomly generated upper triangular matrix of dimension  $m$  is Fishburn when each entry is independently and identically distributed as  $Y$  (except for the upper-left and lower-right corners)?” Our result implies particularly (see (5.2)) that in the primitive case (when  $Y$  is Bernoulli with mean  $\frac{1}{2}$ ), the probability is asymptotic to

$$4 \sum_{k \geq 0} (-1)^k 2^{-\binom{k+1}{2}} \sum_{0 \leq j \leq k} \prod_{1 \leq \ell \leq j} \frac{1}{1 - 2^{-\ell}} \approx 0.33359 \dots$$

In other words, if we fix the two corners on the diagonal of the matrix to be 1, and generate all other entries by throwing an unbiased coin, consistently putting 0 or 1 as the entry according as the coin being head or tail, each independently of the others, then *more than one third of such matrices are Fishburn*.

**1.3. Asymptotic density of connected regular LCDs.** The proof of Theorem 1 is based on the saddle-point approach developed by the first two authors in [14] and a generalization of the Rogers-Fine identity due to Andrews and Jelínek [2], while Theorem 3 follows from a partial fraction decomposition and is simpler in nature.

It turns out that our saddle-point method is also useful in solving a conjecture of Stoimenow [22] that was subsequently reformulated by Zagier [23], where the enumeration of chord diagrams was studied in order to derive an upper bound for the dimension of the Vassiliev invariants space for knots. Based on numerical evidence, an asymptotic relation for the proportion of connected regular LCDs (among all regular LCDs) was then conjectured; see also [6, 23].

**Theorem 4** (A conjecture in [22]). *Let  $f_n$  be the number of regular LCDs of size  $n$  (which equals the  $n$ -th Fishburn number), and  $g_n$  be that of connected regular LCDs of size  $n$ . Then*

$$(1.12) \quad \frac{g_n}{f_n} = e^{-1} (1 + O(n^{-1})).$$

The same limit result also holds for the derangement probability and the proportion of connected (ordinary) chord diagrams, a well-known result; see for example [3, 21] and [20, A068985].

Let  $g(z) := \sum_{n \geq 1} g_n z^n$ . Then the first few terms of  $g(z)$  are given by (see [20, A022494])

$$(1.13) \quad g(z) = z + z^2 + 2z^3 + 5z^4 + 16z^5 + 63z^6 + 293z^7 + 1561z^8 + 9321z^9 + \dots$$

Our proof of (1.12) relies crucially on the functional equation obtained by Zagier in [23]:

$$(1.14) \quad \Phi(z, g(z)) = 1, \quad \text{with} \quad \Phi(z, v) := \frac{1}{1+v} \sum_{k \geq 0} \prod_{1 \leq j \leq k} \frac{1 - (1-z)^j}{1 + v(1-z)^j},$$

together with a generalized Rogers-Fine identity derived by Andrews and Jelínek [2]. The function  $\Phi$  is connected to  $F$  in (1.5) when  $\Lambda(z) = (1-z)^{-1}$  by

$$(1.15) \quad F(z, v) = \frac{1}{v} \Phi\left(z, \frac{1}{v} - 1\right).$$

It is through this connection that our analytic techniques can be applied to solve the conjecture (1.12).

This paper is organized as follows. We prove in the next section Theorem 2 concerning the asymptotics of the expected dimension and the variance. We also sketch briefly the approach we developed in [14]. Then the normal limit law of the dimension (Theorem 1) is established in Section 3, and the corresponding dual version in Section 5. Stoimenow’s conjecture, which is now



our Theorem 4, is confirmed in Section 4. Finally, we describe very briefly in Section 6 the limit results for the dimension in the self-dual case, and the case when  $\lambda_1 = 0$ ,  $\lambda_2 > 0$  and there exists at least one odd number in the entry-set. We conclude by mentioning other possible approximation theorems (convergence rates in the central theorems and local limit theorems).

*Throughout this paper, the generic symbols  $c, \varepsilon > 0$  always denote a constant and small quantity, respectively, whose values may not be the same at each occurrence. In contrast, the pair  $(\mu, \sigma^2)$  always stands for the same value given in (1.6). Furthermore, the notation  $a_n \asymp b_n$  means that the ratio  $a_n/b_n$  remains bounded and not equal to zero as  $n$  tends to infinity.*

## 2. THE MEAN AND THE VARIANCE OF THE DIMENSION

We prove Theorem 2 in this section, together with a few related properties.

**2.1. The generating functions of moments.** Define

$$(2.1) \quad M_h(z) := \sum_{n \geq 0} a_n \mathbb{E}(X_n^h) z^n = \partial_s^h F(z, e^s)|_{s=0} \quad (h = 0, 1, \dots)$$

to be (up to the normalizing factor  $a_n$ ) the generating function of the  $h$ th moment of  $X_n$ , where  $F$  is given in (1.5). In particular,  $M_0(z)$  corresponds to the generating function in (1.3).

**Lemma 5.** *The generating function of the  $h$ th moment of  $X_n$  satisfies*

$$(2.2) \quad M_h(z) = U_h(z)M_0(z) + V_h(z),$$

for  $h \geq 0$ , where  $(\{j^h\})$  are the Stirling numbers of the second kind

$$(2.3) \quad \begin{aligned} U_h(z) &:= \sum_{0 \leq \ell \leq h} \begin{Bmatrix} h+1 \\ \ell+1 \end{Bmatrix} (-1)^{h-\ell} \ell! \prod_{1 \leq j \leq \ell} \frac{1}{1 - \Lambda(z)^{-j}} \\ V_h(z) &:= \sum_{0 \leq \ell \leq h} \begin{Bmatrix} h+1 \\ \ell+1 \end{Bmatrix} (-1)^{h+1-\ell} \ell! \sum_{0 \leq k < \ell} \prod_{k < j \leq \ell} \frac{1}{1 - \Lambda(z)^{-j}}. \end{aligned}$$

*Proof.* By taking the derivative with respect to  $v$  on both sides of (1.5) and then substituting  $v = 1$ , we obtain

$$\begin{aligned} M_1(z) &= \partial_v F(z, v)|_{v=1} = \sum_{k \geq 0} \left( \sum_{1 \leq l \leq k} \Lambda(z)^{-l} \right) \left( \prod_{1 \leq j \leq k} (1 - \Lambda(z)^{-j}) \right) \\ &= \sum_{k \geq 0} \left( -1 + \frac{1 - \Lambda(z)^{-k-1}}{1 - \Lambda(z)^{-1}} \right) \left( \prod_{1 \leq j \leq k} (1 - \Lambda(z)^{-j}) \right). \end{aligned}$$

It follows that

$$(2.4) \quad M_1(z) = \frac{M_0(z) - \Lambda(z)}{\Lambda(z) - 1}.$$

In a similar way,

$$\begin{aligned} M_2(z) &= (\partial_v^2 F(z, v) + \partial_v F(z, v))|_{v=1} \\ &= \sum_{k \geq 0} \left( 1 - 3 \frac{1 - \Lambda(z)^{-k-1}}{1 - \Lambda(z)^{-1}} + 2 \frac{(1 - \Lambda(z)^{-k-1})(1 - \Lambda(z)^{-k-2})}{(1 - \Lambda(z)^{-1})(1 - \Lambda(z)^{-2})} \right) \left( \prod_{1 \leq j \leq k} (1 - \Lambda(z)^{-j}) \right). \end{aligned}$$

Thus

$$\begin{aligned}
(2.5) \quad M_2(z) &= M_0(z) - \frac{3(M_0(z) - 1)}{1 - \Lambda(z)^{-1}} + \frac{2(M_0(z) - 2 + \Lambda(z)^{-1})}{(1 - \Lambda(z)^{-1})(1 - \Lambda(z)^{-2})} \\
&= \frac{1 + 2\Lambda(z) - \Lambda(z)^2}{(\Lambda(z) - 1)(\Lambda(z)^2 - 1)} M_0(z) - \frac{\Lambda(z)(3 - 2\Lambda(z) + \Lambda(z)^2)}{(\Lambda(z) - 1)(\Lambda(z)^2 - 1)}.
\end{aligned}$$

The general form (2.2) is then proved by the same arguments and induction.  $\square$

**2.2. Combinatorial interpretations.** Recall that  $a_n$  and  $\mu_n$  are defined in (1.3) and (1.7), respectively. From (2.4), we have the identity

$$\sum_{1 \leq j < n} \lambda_j a_{n-j} \mu_{n-j} = a_n - \lambda_n \quad (n \geq 1),$$

where  $a_n \mu_n = [z^n] M_1(z)$ . In particular, in the primitive case when  $\Lambda(z) = 1 + z$ , we have a surprisingly simple identity for the expected dimension:

$$\mu_n = \frac{a_{n+1}}{a_n},$$

or, in words, *the expected dimension equals the ratio between the number of primitive FMs of size  $n + 1$  and that of size  $n$ .*

Similarly, for the second moment, we have the identity

$$a_n \mathbb{E}(X_n^2) + 2a_{n+1} \mathbb{E}(X_{n+1}^2) = 2a_{n+3} - a_{n+1}.$$

These simple relations certainly demand for combinatorial interpretations, which are given in the following forms.

**Proposition 6.** *Let  $p_{n,k}$  denote the number of primitive FMs of size  $n$  and dimension  $k$ . Then for  $n \geq 1$*

$$(2.6) \quad a_{n+1} = \sum_{1 \leq k \leq n} k p_{n,k},$$

$$(2.7) \quad a_{n+3} = \sum_{1 \leq k \leq n} \binom{k+1}{2} p_{n,k} + \sum_{1 \leq k \leq n+1} k^2 p_{n+1,k}.$$

Among the diverse Fishburn structures, we find it simpler to interpret (2.6) and (2.7) in the language of ascent sequences, listed in Table 1. We can then translate the recursive construction on primitive ascent sequences into primitive FMs via the bijection in [8].

**Definition 1** (Ascent sequence). Let  $\mathcal{I}_n$  be the set of inversion sequences of length  $n$ , namely,

$$\mathcal{I}_n := \{s = (s_1, s_2, \dots, s_n) : 0 \leq s_j < j, 1 \leq j \leq n\},$$

For any sequence  $s \in \mathcal{I}_n$ , let

$$(2.8) \quad \mathbf{asc}(s) := |\{1 \leq j < n : s_j < s_{j+1}\}|$$

be the number of **ascents** of  $s$ . An *inversion sequence*  $s \in \mathcal{I}_n$  is an *ascent sequence* if for all  $2 \leq j \leq n$ ,  $s_j$  satisfies

$$s_j \leq \mathbf{asc}(s_1, s_2, \dots, s_{j-1}) + 1.$$

An ascent sequence is *primitive* if no consecutive entries are identical.

*Proof.* (Proposition 6) It is known (see [4, 8]) that  $p_{n,k}$  also enumerates the number of primitive ascent sequences with  $k - 1$  ascents. For instance,  $p_{4,3} = 4$ : the corresponding primitive ascent sequences are 0121, 0120, 0102 and 0101 and they are in bijection with the following primitive FMs from left to right, respectively.

$$\begin{pmatrix} 1 & 0 & 0 \\ & 1 & 1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ & 0 & 1 \\ & & 1 \end{pmatrix}$$

Given a primitive ascent sequence  $s$  of length  $n$  and with  $k - 1$  ascents, we add a new entry at the end of  $s$ , which can be any integer from  $[0, k]$  but not equal to the last entry  $s_n$  of  $s$ . In other words, there are  $k$  possible ways to add such an integer so that the resulting sequence is a primitive ascent sequence of length  $n + 1$ , which leads to (2.6).

Now we extend the same proof to show (2.7). Given a primitive ascent sequence  $s$  of length  $n + 1$  and with  $k - 1$  ascents, we add two entries  $x, y$  at the end of  $s$ , where  $0 \leq x \leq k$ ,  $x \neq s_{n+1}$ , and  $0 \leq y \leq k$ ,  $y \neq x$ . That is, there are  $k^2$  possible values for the pair  $(x, y)$ . By Definition 1, the resulting sequence  $s^* = s_1 \dots s_{n+1}xy$  is a primitive ascent sequence of length  $n + 3$  such that if the penultimate entry is removed, the resulting sequence is still an ascent sequence.

On the other hand, given a primitive ascent sequence  $s$  of length  $n$  and with  $k - 1$  ascents, we add three entries  $x, y, z$  at the end of  $s$  so that the resulting sequence  $s^* = s_1 \dots s_nxyz$  is not an ascent sequence if the penultimate entry is removed, i.e.,  $s^*$  satisfies  $x < y < z = \text{asc}(s^*)$ . If  $y = k + 1$ , then  $s_n < x \leq k$  and  $z = k + 2$ , implying that there are  $k - s_n$  possible choices for  $x$ ; otherwise  $0 \leq y \leq k$ . Since  $0 \leq x < y \leq k$  and  $x \neq s_n$ , there are  $\frac{1}{2}k(k - 1) + s_n$  different values for the pair  $(x, y)$  and  $z = k + 1$ . It follows that there are in total  $\frac{1}{2}k(k - 1) + k = \frac{1}{2}k(k + 1)$  choices for  $(x, y)$ , and the resulting sequence  $s^* = s_1 \dots s_nxyz$  is a primitive ascent sequence.

Since any primitive ascent sequence of length  $n + 3$  can be produced by either construction, we thus conclude the identity (2.7).  $\square$

**Remark 2.** When  $\Lambda(z) = (1 - z)^{-1}$ , we have instead the pair of relations

$$\begin{cases} a_{n+1} - a_n = \sum_{1 \leq k \leq n} k \bar{p}_{n,k}, \\ a_{n+3} - a_{n+2} = \sum_{1 \leq k \leq n+1} k(k+2) \bar{p}_{n+1,k} - \sum_{1 \leq k \leq n} \binom{k+1}{2} \bar{p}_{n,k}. \end{cases}$$

Such  $\bar{p}_{n,k}$  denotes the number of size- $n$  FMs of dimension  $k$ ; see [20, A137251]. Similar combinatorial interpretations can be given as in the primitive case.

**2.3. The two-stage saddle-point approach.** For self-containedness and to pave the way for proving the asymptotic normality of the dimension, we sketch here the major steps of the two-stage saddle-point method developed in [14] for (1.3), at the same time also indicating how to obtain a finer asymptotic expansion for  $a_n$ .

**2.3.1.  $q$ -series transformation.** The approach starts from the generating function (1.3), which contains nevertheless terms with negative coefficients in the Taylor expansion of  $1 - \Lambda(z)^{-j}$ , which in turn, after multiplication over  $1 \leq j \leq k$ , results in alternating terms that produce severe cancellations in the final summation; see [14] for more details. Instead of manipulating the heavy

cancellations, it is technically more convenient to work on the right-hand side of the identity

$$(2.9) \quad \sum_{k \geq 0} \prod_{1 \leq j \leq k} (1 - \Lambda(z)^{-j}) = \sum_{k \geq 0} \Lambda(z)^{k+1} \prod_{1 \leq j \leq k} (\Lambda(z)^j - 1)^2,$$

as the right-hand side of (2.9) contains only terms with nonnegative Taylor coefficients. This identity is obtained by applying a  $q$ -identity due to Andrews and Jelínek [2, Proposition 2.3]:

$$(2.10) \quad \begin{aligned} & \sum_{k \geq 0} u^k \prod_{1 \leq j \leq k} \left( 1 - \frac{1}{(1-s)(1-t)^{j-1}} \right) \\ &= \sum_{k \geq 0} (1-s)(1-t)^k \prod_{1 \leq j \leq k} \left( (1 - (1-s)(1-t)^{j-1})(1 - u(1-t)^j) \right), \end{aligned}$$

after substituting  $u = 1$  and  $s = t = 1 - \Lambda(z)$  on both sides.

**2.3.2. The exponential prototype.** From the transformed generating function (2.9), it proves much simpler to work out first the special case when  $\Lambda(z) = e^z$  (because we assume  $\lambda_1 > 0$ ). We will see later how to recover the asymptotics of  $a_n$  in general cases.

Let

$$(2.11) \quad E(z) := \sum_{k \geq 0} E_k(z), \quad \text{with} \quad E_k(z) := e^{(k+1)z} \prod_{1 \leq j \leq k} (e^{jz} - 1)^2,$$

and

$$e_n := \sum_{0 \leq k \leq \lfloor \frac{1}{2}n \rfloor} e_{n,k}, \quad \text{with} \quad e_{n,k} := [z^n] E_k(z) = \frac{1}{2\pi i} \oint_{|z|=r} z^{-n-1} E_k(z) dz,$$

where  $r > 0$ . The sequence  $n!e_n$  is essentially *the number of labelled interval orders on  $n$  points*; see [5] and [20, A079144].

**2.3.3. The saddle-point method.** Define

$$(2.12) \quad I(z) := \int_0^z \frac{t}{1 - e^{-t}} dt = \frac{z^2}{2} + \operatorname{dilog}(e^{-z}),$$

where  $\operatorname{dilog}(z)$  denotes the dilogarithm function

$$\operatorname{dilog}(z) := \int_0^z \frac{\log u}{1 - u} du.$$

The asymptotic analysis of  $e_n$  is then split into the following steps.

- (i) Apply first the saddle-point bound for the Taylor coefficients  $e_{n,k} \leq r^{-n} E_k(r)$ , where  $r > 0$  solves  $nE_k(r) = rE'_k(r)$ , namely,

$$n - (k+1)r = \sum_{1 \leq j \leq k} \frac{2jr}{1 - e^{-jr}} \sim \frac{2I(kr)}{r}.$$

By the asymptotic behaviors of  $I(x)$  as  $x \rightarrow 0$  and  $x \rightarrow \infty$ , we see that such an  $r$  exists as long as  $0 \leq k < \lfloor \frac{1}{2}n \rfloor$  and satisfies  $r \asymp (n - 2k)(k + 1)^{-2}$ . This choice of  $r$  then gives ( $q = k/n$ )

$$(2.13) \quad e_{n,k} = O(n^{n+1} e^{\phi(q, \mu^{-1}n)}), \quad \text{with} \quad \phi(q, \xi) := 2q \log(e^{q\xi} - 1) - 1 - \log \xi,$$

where  $\xi$  is connected to  $q$  by the relation  $2I(q\xi) = \xi$ .

(ii) Find the positive solution pair  $(q, \xi)$  of the equations

$$\partial_q \phi(q, \xi) = 0 \quad \text{and} \quad 2I(q\xi) = \xi,$$

so as to maximize  $\phi(q, \xi)$ . The solution is then given by

$$(2.14) \quad (q, \xi) = (\mu \log 2, \mu^{-1}).$$

(iii) We then further shrink the dominant range to  $|k - qn| \leq n^{\frac{1}{2} + \varepsilon}$ ,  $\varepsilon > 0$ , where most contribution to  $e_n$  will come. It suffices to choose  $\varepsilon = \frac{1}{8}$ , and show, by the saddle-point bound (2.13) and the concavity of  $\phi(q, \xi)$ , that the contribution to  $e_n$  of  $e_{n,k}$  from the range  $|k - qn| \geq n^{\frac{5}{8}}$  is asymptotically negligible.

(iv) In the central range  $|k - qn| \leq n^{\frac{5}{8}}$ , show that the integral

$$\int_{\substack{z=re^{i\theta} \\ n^{-\frac{3}{8}} \leq |\theta| \leq \pi}} z^{-n-1} E_k(z) \, dz$$

is asymptotically negligible. The key property used is the following concentration inequality (see also [14, Lemma 13])

$$(2.15) \quad |E_k(re^{it})| \leq E_k(r) \exp\left(-\frac{(k+1)^2 r t^2}{\pi^2}\right),$$

uniformly for  $k \geq 0$ ,  $r > 0$  and  $|t| \leq \pi$ .

(v) Then inside the ranges  $|k - qn| \leq n^{\frac{5}{8}}$ , compute the integral

$$\int_{\substack{z=re^{i\theta} \\ |\theta| \leq n^{-\frac{3}{8}}}} z^{-n-1} E_k(z) \, dz$$

by more precise local expansions, standard Gaussian approximation, and term-by-term integration, after deriving a fine asymptotic expansion for the saddle-point  $r$ .

(vi) Summing over the asymptotics of  $e_{n,k}$  and approximating the sum by an integral give (1.3).

(vii) Refine steps (iv) and (v) by using a longer expansion if more terms in the asymptotic expansion are desired.

We then obtain not only (1.3) when  $\Lambda(z) = e^z$  but also a refined asymptotic expansion

$$(2.16) \quad \frac{e_n}{cn^{\frac{1}{2}} \mu^n n!} = 1 + \sum_{1 \leq j < j_0} \tilde{d}_j n^{-j} + O(n^{-j_0}),$$

for any  $j_0 = 1, 2, \dots$ , where  $(c, \mu)$  is given in (1.4) when  $\Lambda(z) = e^z$ , and, in particular,

$$\tilde{d}_1 = \frac{3}{8} + \frac{\pi^2}{144}, \quad \text{and} \quad \tilde{d}_2 = -\frac{7}{128} - \frac{\pi^2}{1152} + \frac{\pi^4}{41472}.$$

2.3.4. *From  $e^z$  back to  $\Lambda(z)$ .* To recover the asymptotics (1.3) from the special case when  $\Lambda(z) = e^z$ , we use the following change-of-variables arguments based on the Cauchy integral representation of  $a_n$ :

$$(2.17) \quad a_n := \sum_{0 \leq k \leq \lfloor \frac{1}{2}n \rfloor} [z^n] A_k(z) \quad \text{with} \quad A_k(z) := \Lambda(z)^{k+1} \prod_{1 \leq j \leq k} (\Lambda(z)^j - 1)^2,$$

We then make the change of variables  $\Lambda(z) = e^y$ , which is locally invertible when  $z \sim 0$  because  $\lambda_1 > 0$ , so that  $z = \lambda_1^{-1}y\psi(y)$ , where  $\psi(y)$  satisfies

$$\psi(y) = 1 + \left(\frac{1}{2} - \frac{\lambda_2}{\lambda_1^2}\right)y + \left(\frac{1}{6} - \frac{\lambda_1^2\lambda_2 + \lambda_1\lambda_3 - 2\lambda_2^2}{\lambda_1^4}\right)y^2 + \dots$$

The analyticity of  $\Lambda$  also implies the boundedness of  $\psi$  when  $y$  is small. Here we also see the magic constant “ $\frac{\lambda_2}{\lambda_1^2}$ ” appears in the linear term, which is the source of all the occurrences in the second-order terms in the moments approximations; see (1.8) and (1.9). Then

$$a_n = \lambda_1^n [y^n] \Psi_n(y) = \sum_{0 \leq k \leq \lfloor \frac{1}{2}n \rfloor} E_k(y) = \lambda_1^n [y^n] \Psi_n(y) = \sum_{0 \leq k \leq n} e_k y^k,$$

where  $\Psi_n(y) := \psi(y)^{-n-1}(\psi(y) + y\psi'(y))$ . With  $d := -\frac{1}{2} + \frac{\lambda_2}{\lambda_1^2}$ , we have the asymptotic expansion

$$\Psi_n\left(\frac{t}{n}\right) = e^{dt} \left(1 + \frac{\varpi_1(t)}{n} + \frac{\varpi_2(t)}{n^2} + \dots\right),$$

where the  $\varpi_j$ 's are polynomials in  $t$  of degree  $2j$ . Now expand each term on the right-hand side at  $t = t_0 := \frac{\pi^2}{6}$ , compute the coefficient of  $t^n$  term by term, and then estimate the corresponding error terms; see [14] for details. We then obtain an asymptotic expansion in decreasing powers of  $n$ , which, for easier reference, is stated formally as follows. All steps involved are readily coded (except for the justification ones).

**Proposition 7.** *Assume that  $\Lambda(z)$  is analytic at  $z = 0$  and  $\lambda_1 > 0$ . Then the number of  $\Lambda$ -FMs of size  $n$  satisfies the asymptotic expansion*

$$(2.18) \quad \frac{a_n}{cn^{\frac{1}{2}}(\lambda_1\mu)^n n!} = 1 + \sum_{1 \leq j < j_0} d_j n^{-j} + O(n^{-j_0}),$$

for any  $j_0 = 1, 2, \dots$ , where  $(c, \mu)$  is given in (1.4), and, in particular,

$$(2.19) \quad d_1 = \frac{3}{8} + \frac{19\lambda_1^2 - 36\lambda_2}{144\lambda_1^2} \pi^2 + \frac{\lambda_1^2 + 12\lambda_1\lambda_3 - 12\lambda_2^2}{432\lambda_1^4} \pi^4,$$

$$d_2 = -\frac{7}{128} - \frac{(19\lambda_1^2 - 36\lambda_2)\pi^2}{1152\lambda_1^2} - \frac{(35\lambda_1^4 + 456\lambda_1^2\lambda_2 + 1872\lambda_1\lambda_3 - 2304\lambda_2^2)\pi^4}{41472\lambda_1^4}$$

$$+ \frac{(7\lambda_1^6 - 12\lambda_1^4\lambda_2 + 228\lambda_1^3\lambda_3 - 228\lambda_1^2\lambda_2^2 + 288\lambda_1^2\lambda_4 - 1008\lambda_1\lambda_2\lambda_3 + 720\lambda_2^3)\pi^6}{62208\lambda_1^6}$$

$$- \frac{(5\lambda_1^4 - 12\lambda_1^2\lambda_2 + 24\lambda_1\lambda_3 - 12\lambda_2^2)(\lambda_1^4 - 12\lambda_2\lambda_1^2 - 24\lambda_1\lambda_3 + 36\lambda_2^2)\pi^8}{1492992\lambda_1^8}.$$

See [14] for an alternative approach to (2.18), based on Zagier's approach (which in turn relies on other identities and quantum modular forms).

We list the expressions of  $d_1$  and  $d_2$  in the two standard cases of FMs:

	$\Lambda(z) = (1 - z)^{-1}$	$\Lambda(z) = 1 + z$
$d_1$	$\frac{3}{8} - \frac{17\pi^2}{144} + \frac{\pi^4}{432}$	$\frac{3}{8} + \frac{19\pi^2}{144} + \frac{\pi^4}{432}$
$d_2$	$-\frac{7}{128} + \frac{17\pi^2}{1152} - \frac{59\pi^4}{41472} - \frac{5\pi^6}{62208} - \frac{5\pi^8}{1492992}$	$-\frac{7}{128} - \frac{19\pi^2}{1152} - \frac{35\pi^4}{41472} + \frac{7\pi^6}{62208} - \frac{5\pi^8}{1492992}$

In particular, the expression  $d_1$  is consistent with the expression given in [23, p. 955].

**2.4. Asymptotics of the moments.** With the expansion (2.18) available, we are now ready to derive the asymptotics of the first two moments and prove Theorem 2.

By (2.4), we have, as  $z \sim 0$ ,

$$M_1(z) = \frac{M_0(z) - \Lambda(z)}{\Lambda(z) - 1} = \left( \frac{1}{\lambda_1 z} - \frac{\lambda_2}{\lambda_1^2} + O(|z|) \right) M_0(z) + O(1).$$

Then

$$\mathbb{E}(X_n) = \frac{[z^n]M_1(z)}{a_n} = \frac{a_{n+1}}{\lambda_1 a_n} - \frac{\lambda_2}{\lambda_1^2} + O\left(\frac{a_{n-1}}{a_n}\right),$$

which, together with (2.18), gives

$$\mathbb{E}(X_n) = \frac{a_{n+1}}{\lambda_1 a_n} - \frac{\lambda_2}{\lambda_1^2} + O(n^{-1}) = \mu n + \frac{9}{\pi^2} - \frac{\lambda_2}{\lambda_1^2} + O(n^{-1}).$$

This proves (1.8), the first part of Theorem 2. Note that with the weaker form (1.3), the constant term cannot be made explicit. Further terms can be readily computed by computer algebra software; for example, using the expression of  $d_1$  in (2.19),

$$\mathbb{E}(X_n) = \mu \left( n + \frac{3}{2} \right) - \frac{\lambda_2}{\lambda_1^2} + \frac{1}{n} \left( \frac{1}{2\pi^2} - \frac{19}{24} - \frac{\pi^2}{72\lambda_1^2} + \frac{3\lambda_2}{2\lambda_1^2} - \frac{2\pi^2\lambda_3}{3\lambda_1^3} + \frac{2\pi^2\lambda_2^2}{3\lambda_1^4} \right) + O(n^{-2}).$$

Similarly, by (2.5),

$$M_2(z) = \left( \frac{1}{\lambda_1^2 z^2} - \frac{\lambda_1^2 + 4\lambda_2}{2\lambda_1^3 z} - \frac{\lambda_1^4 - 2\lambda_1^2\lambda_2 + 8\lambda_1\lambda_3 - 12\lambda_2^2}{4\lambda_1^4} + O(|z|) \right) M_0(z),$$

and accordingly

$$\begin{aligned} \mathbb{V}(X_n) &= \frac{[z^n]M_2(z)}{a_n} - \mu_n^2 \\ &= \frac{a_{n+2}}{\lambda_1^2 a_n} - \frac{(\lambda_1^2 + 4\lambda_2)a_{n+1}}{2\lambda_1^3 a_n} - \frac{\lambda_1^4 - 2\lambda_1^2\lambda_2 + 8\lambda_1\lambda_3 - 12\lambda_2^2}{4\lambda_1^4} + O\left(\frac{a_{n-1}}{a_n}\right) - \mu_n^2. \end{aligned}$$

By the expansion (2.18) and (1.8), we then obtain (1.9) by straightforward calculations. A longer expansion is also easily computed; for example,

$$\mathbb{V}(X_n) = \sigma^2 \left( n + \frac{3}{2} \right) - \frac{1}{4} + \frac{\lambda_2}{2\lambda_1^2} + \frac{1}{n} \left( \frac{\pi^2}{48} - \frac{1}{4\pi^2} + \frac{19}{48} + \frac{\pi^2}{144\lambda_1^2} - \frac{3\lambda_2}{4\lambda_1^2} + \frac{\pi^2 + 12}{6} \left( \frac{\lambda_3}{\lambda_1^3} - \frac{\lambda_2^2}{\lambda_1^4} \right) \right) + O(n^{-2}).$$

### 3. DIMENSION OF RANDOM FMS

This section is devoted to a proof of Theorem 1, the central limit theorem for the dimension of random  $\Lambda$ -FMs of large size.

**3.1. A better bivariate generating function.** We begin with seeking a series representation of  $F(z, v)$  better than (1.5) because (1.5) contains negative coefficients in the expansion of  $1 - \Lambda(z)^{-j}$ . In addition to (1.5), it is also known that (see [12, 15])

$$(3.1) \quad F(z, v) = 1 + \sum_{k \geq 1} \frac{v \Lambda(z)^{-k}}{1 - v(1 - \Lambda(z)^{-k})} \prod_{1 \leq j \leq k} (1 - \Lambda(z)^{-j}),$$

but again the same sign problem occurs. A better expression for our purposes is the following one.

**Lemma 8.** *The bivariate generating function for the dimension of  $\Lambda$ -FMs satisfies*

$$(3.2) \quad F(z, v) = \sum_{n \geq 0} P_n(v) z^n = 1 - v + \sum_{k \geq 1} v^k \Lambda(z)^k \prod_{1 \leq j < k} \frac{(\Lambda(z)^j - 1)^2}{1 - (v-1)(\Lambda(z)^j - 1)},$$

where  $P_n(v)$  is the generating polynomial of dimension of  $\Lambda$ -FMs of size  $n$ .

*Proof.* Substitute  $t = 1$ ,  $s = \frac{v-1}{v(1-z)}$ ,  $x = y = 1 - \Lambda(z)$  in the following identity of Andrews and Jelínek in [2]:

$$(3.3) \quad \sum_{n \geq 0} \frac{\left(\frac{s}{t(1-x)}; 1-x\right)_n \left(\frac{1}{1-y}; \frac{1}{1-x}\right)_n}{(s; 1-x)_n} t^n \\ = (1-y) \sum_{n \geq 0} \frac{(1-y; 1-x)_n (t(1-x); 1-x)_n}{(s; 1-x)_n} (1-x)^n,$$

where  $(a; z)_n := (1-a)(1-az) \cdots (1-az^{n-1})$ . The left-hand side gives (3.1), and the right-hand side (3.2).  $\square$

Interestingly, this lemma gives a combinatorial interpretation of a special case of the generalized Rogers-Fine identity (3.3), partially answering a question raised by Andrews and Jelínek [2].

**3.2. The exponential prototype.** As indicated above, we focus first on the special case when  $\Lambda(z) = e^z$ , the general case being then deduced by an argument based on change of variables.

Let

$$(3.4) \quad E(z, v) = 1 - v + \sum_{k \geq 0} E_k(z) R_k(z, v),$$

where  $E_k(z)$  is defined in (2.11) and

$$(3.5) \quad R_k(z, v) := v^{k+1} \prod_{1 \leq j \leq k} \frac{1}{1 - (v-1)(e^{jz} - 1)} = v \prod_{1 \leq j \leq k} \frac{1}{1 - (1-v^{-1})e^{jz}}.$$

The first few terms in the Taylor expansion of  $E(z, v)$  are

$$E(z, v) = 1 + vz + (v + 2v^2) \frac{z^2}{2!} + (v + 12v^2 + 6v^3) \frac{z^3}{3!} + (v + 50v^2 + 132v^3 + 24v^4) \frac{z^4}{4!} + \cdots$$

Here the coefficient of  $\frac{z^n v^k}{n!}$  counts the number of labelled  $(2+2)$  free posets of  $n$  elements and with magnitude  $k-1$ .

While the Taylor expansion of  $R_k(z, v)$  (in  $z$  and  $v$ ) still contains, in general, negative coefficients, the series (3.5) is suitable for our purposes because  $R_k$  plays asymptotically only a perturbative role when  $v$  is close to 1 in view of the estimate

$$(3.6) \quad R_k(z, v) = \prod_{1 \leq j \leq k} \frac{1}{1 + O(|v-1|)} = e^{O(k|v-1|)},$$

for  $1 \leq k = O(n)$  and small  $z \asymp n^{-1}$  when  $\lambda_1 > 0$ , while, in the same setting,

$$E_k(z) = e^{(k+1)z} \prod_{1 \leq j \leq k} (e^{jz} - 1)^2 = e^{\Omega(k)};$$

see below for more precise analysis.



Our aim is to prove the asymptotic normality of the random variable  $X_n$ , which, in the case of (3.4), is defined as

$$(3.7) \quad \mathbb{P}(X_n = k) := \frac{n! [z^n v^k] E(z, v)}{n! [z^n] E(z, 1)} = \frac{[z^n v^k] E(z, v)}{[z^n] E(z, 1)} \quad (1 \leq k \leq n),$$

for  $n \geq 1$ , and  $X_n$  assumes only integer values. For that purpose, we will restrict our analysis to the range  $|v - 1| \leq \varepsilon$ ,  $v \in \mathbb{C}$ . Then, according to the approach sketched in § 2.3 for the asymptotics of  $e_n$ , we would expect, when  $|v - 1| \leq \varepsilon$ , that

$$\begin{aligned} [z^n] E(z, v) &= [z^n] \sum_{0 \leq k \leq \lfloor \frac{1}{2}n \rfloor} E_k(z) R_k(z, v) \approx \sum_{0 \leq k \leq \lfloor \frac{1}{2}n \rfloor} R_k(r, v) [z^n] E_k(z) \\ &\approx R_{\lfloor qn \rfloor}((\mu n)^{-1}, v) [z^n] E(z, 1), \end{aligned}$$

where  $q = \mu \log 2$ . This is, up to the leading constant, correct because

$$R_{\lfloor qn \rfloor}((\mu n)^{-1}, e^{i\theta/\sqrt{n}}) \sim e^{\mu\sqrt{n}i\theta - \frac{3}{2\pi^2}\theta^2},$$

and we see that while the coefficient of  $i\theta$  matches that of the mean, the coefficient of  $\theta^2$  is not equal to  $-\frac{1}{2}\sigma^2$ , as desired, showing that a more delicate analysis is required.

**Proposition 9.** *For large  $n$ , the coefficient of  $z^n$  in the Taylor expansion of  $E(z, v)$  defined in (3.4) satisfies*

$$(3.8) \quad [z^n] E(z, e^{i\theta/\sqrt{n}}) = cn^{\frac{1}{2}} \mu^n n! \exp\left(\mu\sqrt{n}i\theta - \frac{\sigma^2\theta^2}{2}\right) (1 + O((|\theta| + |\theta|^3)n^{-\frac{1}{2}})),$$

uniformly for  $\theta = o(n^{\frac{1}{6}})$ , where  $c = \frac{12\sqrt{3}}{\pi^{5/2}}$  and  $(\mu, \sigma^2)$  is defined in (1.6).

The approximation (3.8) implies, by (3.7) ( $\mu_n := \mathbb{E}(X_n)$  and  $\sigma_n^2 := \mathbb{V}(X_n)$ ),

$$(3.9) \quad \mathbb{E}\left(\exp\left(\frac{X_n - \mu_n}{\sigma_n} i\theta\right)\right) \rightarrow e^{-\frac{1}{2}\theta^2},$$

uniformly for  $\theta = O(1)$ , and then the asymptotic normality of  $X_n$  (1.6) (when  $\Lambda(z) = e^z$ ) follows from standard convergence theorem for characteristic functions; see, e.g., [11, p. 777].

Throughout this section,  $v \in \mathbb{C}$  always lies in a small neighborhood of unity,  $|v - 1| \leq \varepsilon$ , unless otherwise indicated.

**3.3. The factorial growth order.** The approach we adopt here follows mostly that sketched above in Section 2.3.3 from [14] but is carried out differently, with a particular attempt to keep it more self-contained. We begin with the following lemma, showing that a simple inequality is already sufficient to characterize the factorial growth of the problem; furthermore, it shows that the sum of  $[z^n] E_k(z) R_k(z, v)$  over the ranges  $k \leq \alpha_- n$  and  $k \geq \alpha_+ n$  is asymptotically negligible, where  $\alpha_{\pm}$  are specified below.

**Proposition 10.** *Let  $\alpha_{\pm} > 0$  be the two zeros of the equation  $\varphi(\alpha) = \log \mu$ , where*

$$(3.10) \quad \varphi(\alpha) := 2(1 - \alpha) \log \alpha - (1 - 2\alpha) \log(1 - 2\alpha) + 2 - 4\alpha.$$

Then

$$(3.11) \quad \left( \sum_{0 \leq k \leq (\alpha_- - \varepsilon)n} + \sum_{k \geq (\alpha_+ + \varepsilon)n} \right) [z^n] E_k(z) R_k(z, v) = O(n!(\mu - \varepsilon)^n),$$

uniformly for  $|v - 1| = o(1)$ .

*Proof.* Since the Taylor expansion of  $E_k(z)$  contains only positive coefficients, we have, by the elementary inequality  $e^x - 1 \leq xe^x$  for  $x \geq 0$ ,

$$e_{n,k} \leq r^{-n} E_k(r) = r^{-n} e^{(k+1)r} \prod_{1 \leq j \leq k} (jre^{jr})^2 = r^{-n} e^{(k+1)^2 r} k!^2 r^{2k} \quad (r > 0).$$

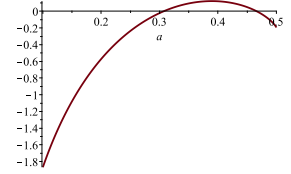
The optimal choice of  $r > 0$  at which the right-hand side reaches its minimum value for fixed  $n$  and  $k$  is obtained by taking derivative with respect to  $r$ , setting it equal to zero and then solving for  $r$ . In this way, we find that the minimum such  $r$  is  $r = r_0 = (n - 2k)(k + 1)^{-2}$ , and we get, with  $k = \alpha n$ ,  $0 < \alpha < \frac{1}{2}$ ,

$$(3.12) \quad e_{n,k} \leq r_0^{-n} E_k(r_0) = O(kn^n e^{(\varphi(\alpha)-1)n}),$$

by Stirling's formula, where  $\varphi$  is defined in (3.10). When  $\alpha = \frac{1}{2}$  or  $n - 2k = \ell = o(n)$ , we take  $r = (1 + \ell)n^{-2}$ , giving the estimate  $e_{n,k} = O(n^{n+2\ell}(2e)^{-n} n^{2\ell} (\ell + 1))^{-2\ell}$ . Then

$$e^{\varphi(\alpha)} < \mu \quad \text{if and only if} \quad 0 \leq \alpha < \alpha_- \quad \text{and} \quad \alpha_+ < \alpha \leq \frac{1}{2},$$

where  $\alpha_{\pm} > 0$  solves the equation  $\varphi(\alpha) = \log \mu$ . Numerically,  $\alpha_- \approx 0.30686$  and  $\alpha_+ \approx 0.46628$ . On the other hand, when  $|v - 1| = o(1)$ , we have, by (3.6) and (3.12),



A plot of  $\varphi(\alpha) - \log \mu$ .

$$[z^n]E_k(z)R_k(z, v) = O(\max_{|z|=r} |R_k(z, v)| r^{-n} E_k(r)) = O(kn^n e^{(\varphi(\alpha)-1+o(1))n}).$$

It follows that

$$\left( \sum_{0 \leq k \leq (\alpha_- - \varepsilon)n} + \sum_{k \geq (\alpha_+ + \varepsilon)n} \right) [z^n]E_k(z)R_k(z, v) = O(n! n^{\frac{3}{2}} (e^{\varphi(\alpha_- - \varepsilon)n} + e^{\varphi(\alpha_+ + \varepsilon)n})).$$

This proves (3.11) since  $\varphi(\alpha_{\pm}) = \log \mu$  and  $\varphi(\alpha)$  is concave on  $[0, \frac{1}{2}]$ .  $\square$

With the estimate (3.11) available, we will limit our asymptotic study of  $[z^n]E_k(z)R_k(z, v)$  to only linear  $k$ , namely  $\varepsilon n \leq k \leq (\frac{1}{2} - \varepsilon)n$ .

**3.4. The exponential growth order.** In this section, we derive the exponential term in the growth rate of  $[z^n]E(z, v)$ , starting from another simple (finer) approximation of  $\log E_k(r)$ . We also establish the asymptotic negligibility of  $[z^n]E_k(z)R_k(z, v)$  for  $k$  outside  $[k_-, k_+]$ , where  $k_{\pm}$  is defined below in Proposition 12.

**Lemma 11.** For  $0 \leq k \leq \lfloor \frac{1}{2}n \rfloor$ ,

$$(3.13) \quad \begin{cases} \log E_k(r) \leq 2k \log(e^{kr} - 1) - 2 \log(e^r - 1) - \int_0^k \frac{2xr}{1 - e^{-xr}} dx + O(1 + kr), \\ \log R_k(z, e^{i\theta}) = \frac{e^{kz} - 1}{z} (1 - e^{-i\theta}) + \frac{e^{2kz} - 1}{4z} (1 - e^{-i\theta})^2 + O(|z|^{-1}|\theta|^3 + |\theta| + |z|), \end{cases}$$

uniformly for  $|z| = r = O(n^{-1})$ , and  $\theta = o(1)$ .

*Proof.* By the monotonicity of  $\log(e^x - 1)$ ,

$$\log E_k(r) = (k + 1)r + 2 \sum_{1 \leq j \leq k} \log(e^{jr} - 1) \leq 2 \int_1^k \log(e^{xr} - 1) dx + O(1 + kr),$$

which then proves the upper bound of  $\log E_k(r)$  in (3.13) by an integration by parts. Note that the integral in (3.13) equals  $2r^{-1}I(kr)$ ; see (2.12).

The other approximation in (3.13) is obtained by the expansion

$$R_k(z, e^{i\theta}) = e^{i\theta} \prod_{1 \leq j \leq k} \frac{1}{1 - (1 - e^{-i\theta})e^{jz}} = \exp\left(i\theta + \sum_{l \geq 1} \frac{(1 - e^{-i\theta})^l}{l} \cdot \frac{e^{klz} - 1}{1 - e^{-lz}}\right). \quad \square$$

**Proposition 12.** With  $q = \mu \log 2 \approx 0.42138$  and  $k_{\pm} := qn \pm 2\zeta n^{\frac{5}{8}}$ , where

$$\zeta^2 := \frac{3(24(\log 2)^2 - \pi^2)}{2\pi^4},$$

we have

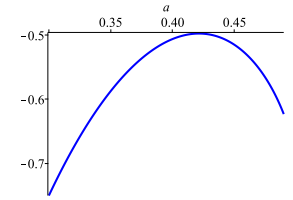
$$(3.14) \quad \left( \sum_{0 \leq k \leq k_-} + \sum_{k_+ \leq k \leq 0.5n} \right) [z^n] E_k(z) R_k(z, e^{i\theta}) = O(n! \mu^n e^{-n} n^2 e^{-\varepsilon n \theta^2 - n^{\frac{1}{4}}}),$$

uniformly for  $\theta = o(1)$ .

*Proof.* By (3.13), we take  $r$  to be the positive solution of the equation  $2I(kr) = rn$ , which exists as long as  $0 \leq k < \lfloor \frac{1}{2}n \rfloor$  by the asymptotic behaviors of  $I(x)$  for small and large  $x$ , and satisfies  $r \asymp (n - 2k)k^{-2}$ ; see [14]. Then, with  $k = \alpha n$  and  $r = \xi n^{-1}$ , we have

$$r^{-n} E_k(r) = O(n^2 e^{-n \log r + 2k \log(e^{kr} - 1) - n}) = O(n^{n+2} e^{\phi(\alpha, \xi)n}),$$

where  $\phi(\alpha, \xi)$  is defined in (2.13) subject to the condition  $2I(\alpha\xi) = \xi$ . The maximum value of  $\phi(\alpha, \xi)$  is characterized by computing the solution of the equation  $\partial_{\alpha} \phi(\alpha, \xi) = 0$ , which is reached at  $(\alpha, \xi) = (\mu \log 2, \mu^{-1})$ . Then we obtain  $\phi(\mu \log 2, \mu^{-1}) = \log \mu - 1$ . In addition, it is easy to prove the concavity of  $\phi(\alpha, \xi)$  when viewed as a function of  $\alpha$ ; see [14, Lemma 11].



A plot of  $\phi(\alpha, \xi) + 1$ ;  
 $\log \mu \approx -0.4977$ .

Now with the estimates in (3.13) and the inequality (2.15), we obtain

$$(3.15) \quad [z^n] E_k(z) R_k(z, v) = O\left(r^{-n} E_k(r) \int_{-\pi}^{\pi} |R_k(re^{it}, v)| \exp\left(-\frac{(k+1)^2 r t^2}{\pi^2}\right) dt\right),$$

where  $r > 0$  is chosen to be the same as above, namely,  $2I(kr) = rn$ . Since

$$\log |R_k(z, e^{i\theta})| = O(|\theta|r^{-1}) = o((k+1)^2 r),$$

when  $\theta = o(1)$ ,  $k \asymp n$  and  $r \asymp n^{-1}$ , and the integral has the typical form amenable to the Laplace's method, we then deduce, by using the local expansion

$$(3.16) \quad \log |R_k(re^{it}, e^{i\theta})| = -\frac{(e^{kr} - 1)^2}{4r} \theta^2 - r \partial_r \left( \frac{e^{kr} - 1}{r} \right) \theta t + O(r^{-1}(\theta^4 + \theta^2 t^2 + |t||\theta|^3)),$$

that

$$(3.17) \quad \int_{-\pi}^{\pi} |R_k(re^{it}, v)| \exp\left(-\frac{(k+1)^2 r t^2}{\pi^2}\right) dt = O\left(\frac{e^{-c'(1+o(1))n\theta^2}}{(k+1)\sqrt{r}}\right),$$

where

$$c' := \frac{(e^{kr} - 1)^2}{r} - \frac{\pi^2(e^{2kr}(kr - 1)^2 + 2e^{kr}(kr - 1) + 1)}{4(k + 1)^2 r^3}.$$

Now, for  $k \sim qn$  and  $r \sim \mu n^{-1}$ , we get

$$c' \sim \frac{-3(\log 2)^2 + 4 \log 2 - 1}{23(\log 2)^2} \approx 0.2835.$$

We next improve on the growth order of  $r^{-k}E_k(r)$  when  $k \sim qn$ . Write

$$(3.18) \quad k = qn + \varsigma\sqrt{n}x \quad (x = o(\sqrt{n})).$$

Solving the equation  $2I(q\xi) = \xi$  for  $\xi$  gives the expansion  $r = \xi n^{-1}$ , where

$$\xi = \frac{1}{\mu} - \frac{\log 2}{\sigma\sqrt{n}}x + \frac{4\pi^2\sigma^2(3 + 2\pi^2\sigma^2) - 9(1 - \log 2)}{48\pi^2\sigma^4n}x^2 + O(|x|^3n^{-\frac{3}{2}}).$$

With these expansions, we then obtain

$$\phi(\alpha, \xi) = \log \mu - \frac{x^2}{2} + \frac{\pi^4 - 72\pi^2 \log 2 + 1152(\log 2)^3}{16\pi^6\sigma^3\sqrt{n}}x^3 + O(x^4n^{-1}).$$

This implies that as long as  $k$  satisfies (3.18), we have

$$r^{-n}E_k(r) = O(n^{n+2}\mu^n e^{-n-\frac{1}{2}x^2+O(|x|^3n^{-\frac{1}{2}})}),$$

uniformly for  $x = o(\sqrt{n})$ . Combining this with the estimates (3.15) and (3.17), we then have

$$|[z^n]E_k(z)R_k(z, e^{i\theta})| = O(n^{n+\frac{3}{2}}\mu^n e^{-n-\varepsilon n\theta^2-\frac{1}{2}x^2(1+o(1))}),$$

uniformly for  $x = o(\sqrt{n})$ . Thus, by the monotonicity of  $\alpha \mapsto \phi(\alpha, \xi)$ , we obtain (3.14).  $\square$

**3.5. The asymptotic equivalent and the proof of Proposition 9.** We complete the proof of Proposition 9 in this section. The analysis is similar to that conducted in the proof of Proposition 12, and will be brief.

We first derive a more precise approximation to  $\log E_k(z)R_k(z, v)$  than (3.13) for  $v$  close to 1.

**Lemma 13.** *For  $z \in \mathbb{C}$ ,  $z \neq 0$ , we have*

$$(3.19) \quad \begin{aligned} \log E_k(z)R_k(z, e^{i\theta}) &= 2k \log(e^{kz} - 1) - \frac{2I(kz)}{z} + \frac{(e^{kz} - 1)i\theta}{z} + \log \frac{2\pi(e^{kz} - 1)}{z} \\ &\quad + kz - \frac{(e^{kz} - 1)^2\theta^2}{4z} + O(k^{-1} + |\theta| + n|\theta|^3), \end{aligned}$$

uniformly for  $\theta = o(1)$ ,  $|z| \asymp n^{-1}$  and  $1 \leq k \leq \lfloor \frac{1}{2}n \rfloor$ .

*Proof.* The expansion (3.19) follows from the Euler-Maclaurin formula.  $\square$

We also need the following estimate for the tail of a Gaussian integral.

**Lemma 14.** *Assume  $K, t_1 > 0$  and  $L \in \mathbb{R}$ . Then*

$$(3.20) \quad \int_{t_1}^{\infty} e^{-Lt - Kt^2} dt = O(e^{-t_1(Kt_1 + L)}(2Kt_1 + L)^{-1}),$$

as long as  $(2Kt_1 + L)K^{-\frac{1}{2}} \rightarrow \infty$ .

*Proof.* By a direct change of variables  $u = Lt + Kt^2 + \frac{L^2}{4K}$ ,

$$\int_{t_1}^{\infty} e^{-Lt - Kt^2} dt = \frac{e^{L^2/(2K)}}{2\sqrt{K}} \int_{t_2}^{\infty} e^{-u} u^{-\frac{1}{2}} du,$$

where  $t_2 := (2Kt_1 + L)^2(4K)^{-1}$ . Now we have, as  $t_2 \rightarrow \infty$ ,

$$\int_{t_2}^{\infty} e^{-u} u^{-\frac{1}{2}} du \sim t_2^{-\frac{1}{2}} e^{-t_2}. \quad \square$$

Now we are ready to complete the proof of Proposition 9.

In what follows, write  $v = e^{i\theta/\sqrt{n}}$ . Then

$$[z^n]E_k(z)R_k(z, e^{i\theta/\sqrt{n}}) = \frac{r^{-n}}{2\pi} \left( \int_{|t| \leq t_0} + \int_{t_0 < |t| \leq \pi} \right) e^{-int} E_k(re^{it})R_k(re^{it}, e^{i\theta/\sqrt{n}}) dt,$$

where  $t_0 := 6n^{-\frac{3}{8}}$ . Since  $t_0$  is small, the second integral is estimated by the same arguments used above for (3.15), and we are led to an integral of the form (3.20) with

$$K = \frac{(k+1)^2 r}{\pi^2}, \quad L = \frac{r\theta}{\sqrt{n}} \partial_r \left( \frac{e^{kr} - 1}{r} \right), \quad \text{and} \quad t_1 = t_0.$$

Since

$$\frac{Kt_1 + L}{\sqrt{K}} \sim \frac{6\sqrt{6} \log 2}{\pi^2} n^{\frac{1}{8}} > n^{\frac{1}{8}},$$

when  $k \sim qn$  and  $r \sim (\mu n)^{-1}$ , we have, by (3.15), (3.16) and (3.20),

$$\begin{aligned} & r^{-n} \int_{t_0 < |t| \leq \pi} e^{-int} E_k(re^{it})R_k(re^{it}, e^{i\theta/\sqrt{n}}) dt \\ &= O\left( r^{-n} E_k(r) \int_{t_0}^{\pi} |R_k(re^{it}, e^{i\theta/\sqrt{n}})| \exp\left(-\frac{(k+1)^2 r t^2}{\pi^2}\right) dt \right) \\ &= O\left( r^{-n} E_k(r) (k+1)^{-1} r^{-\frac{1}{2}} e^{-\varepsilon \theta^2 - n^{\frac{1}{4}}}\right), \end{aligned}$$

which will be seen to be negligible.

Denote by

$$\Xi_2(r) := 2[s^2] \log(E_k(re^s)R_k(re^s, e^{i\theta/\sqrt{n}})),$$

where  $r > 0$  solves the saddle-point equation  $[s] \log E_k(re^s)R_k(re^s, e^{i\theta/\sqrt{n}}) = n$ . The analysis we carried out so far implies that the saddle-point approximation

$$(3.21) \quad [z^n]E_k(z)R_k(z, e^{i\theta/\sqrt{n}}) = \frac{r^{-n} E_k(r) R_k(r, e^{i\theta/\sqrt{n}})}{\sqrt{2\pi \Xi_2(r)}} \left( 1 + O(\Xi_2(r)^{-1}) \right),$$

is well-justified for  $k_- \leq k \leq k_+$ , provided that  $\Xi_2(r) \rightarrow \infty$ , which will be seen to be the case. Here  $k_{\pm}$  is defined in Proposition 12.

By the Euler-Maclaurin formula or simply (3.19), the saddle-point equation is given asymptotically by

$$\begin{aligned} \frac{I(kr)}{r} - n + \frac{e^{kr}(kr-1) + 1}{r\sqrt{n}} i\theta \\ + \frac{1}{rn} \left( \frac{kr^2n}{e^{kr}-1} + rn(2kr-1) - \frac{(e^{kr}-1)(e^{kr}(2kr-1) + 1)\theta^2}{4} \right) + O(n^{-1}) = 0. \end{aligned}$$

The remaining steps are readily coded, and we sketch only the main steps.

Write  $k = qn + \varsigma\sqrt{n}x$ . Assuming an expansion of the form

$$(3.22) \quad r = \sum_{j \geq 0} r_j n^{-\frac{1}{2}j-1},$$

where  $r_0 = \mu^{-1}$ , we then derive, by a standard bootstrapping argument, that

$$r_1 = -\frac{\log 2}{\varsigma} x - \frac{3(2\log 2 - 1)}{2\pi^2\varsigma^2} i\theta,$$

and  $r_j$  is a polynomial of  $x$  and  $\theta$  of degree  $j$  (the expressions are very messy for  $j \geq 2$ ). Substituting this expansion and  $k = qn + \varsigma\sqrt{n}x$  into (3.21) using the asymptotic approximation (3.19), we then deduce that

$$\frac{r^{-n} E_k(r) R_k(r, e^{i\theta/\sqrt{n}})}{c_0 \mu^n e^{-n} n^{n+1} e^{\mu\sqrt{n}i\theta - \frac{1}{2}\sigma^2\theta^2}} = e^{-\frac{1}{2}\left(x + \frac{3(\pi^2 - 12\log 2)i\theta}{\pi^4\varsigma}\right)^2} \left(1 + O\left(\frac{|x| + |\theta| + (|x| + |\theta|)^3}{\sqrt{n}}\right)\right),$$

uniformly for  $x, \theta = o(n^{\frac{1}{6}})$ , where  $c_0 = 24\pi^{-1}$ .

On the other hand,

$$\Xi_2(r) = \frac{e^{kr}(2k^2r - n) + n}{e^{kr} - 1} + \frac{k^2 r e^{kr}}{\sqrt{n}} i\theta + O(1),$$

or, by substituting the expansions of  $k$  and  $r$ ,

$$\Xi_2(r) = \frac{2\pi^2\varsigma^2}{3} n + O((|x| + |\theta|)\sqrt{n}).$$

Collecting these expansions to (3.21) yields

$$\frac{[z^n] E_k(z) R_k(z, e^{i\theta/\sqrt{n}})}{c \mu^n e^{-n} n^{n+\frac{1}{2}} e^{\mu\sqrt{n}i\theta - \frac{1}{2}\sigma^2\theta^2}} = e^{-\frac{1}{2}\left(x + \frac{3(\pi^2 - 12\log 2)i\theta}{\pi^4\varsigma}\right)^2} \left(1 + O\left(\frac{|x| + |\theta| + (|x| + |\theta|)^3}{\sqrt{n}}\right)\right),$$

where  $c = 12\sqrt{3}\pi^{-\frac{5}{2}}$ , as in Proposition 9.

Summing over  $k_- \leq k \leq k_+$  and approximating the sum by a Gaussian integral, we then produce an extra factor of the form

$$\sqrt{2\pi} \varsigma \sqrt{n} (1 + O((|\theta| + |\theta|^3)n^{-\frac{1}{2}})).$$

By Stirling's formula for the factorial, we obtain (3.8), which completes the proof of Proposition 9.

**3.6. Proof of Theorem 1.** We now translate the asymptotic approximation (3.8) for  $[z^n]E(z, v)$  when  $\Lambda(z) = e^z$  into that for  $[z^n]F(z, v)$  (see (3.1)) in the more general situations with  $\lambda_1 > 0$ . By (3.2),

$$P_n(v) := [z^n]F(z, v) = [z^n] \sum_{0 \leq k \leq \lfloor \frac{1}{2}n \rfloor} v \Lambda(z) \prod_{1 \leq j \leq k} \frac{\Lambda(z)(\Lambda(z)^j - 1)^2}{1 - (1 - v^{-1})\Lambda(z)^j}.$$

Since  $\Lambda(z) = 1 + \lambda_1 z + \dots$  with  $\lambda_1 \neq 0$  is analytic at  $z = 0$ , the function is locally invertible at  $z = 0$  and we can make the change of variables  $\Lambda(z) = e^y$ , namely, let  $z = \beta(y)$  so that  $\Lambda(\beta(y)) = e^y$  ( $\beta(y)$  also analytic at  $y = 0$ ). Then we have

$$P_n(y) = [y^n]\Psi_n(y, v) \sum_{0 \leq k \leq \lfloor \frac{1}{2}n \rfloor} E_k(y)R_k(y, v),$$

where  $\Psi_n(y) := \beta(y)^{-n-1}y^n\beta'(y)$ . Now the expansion

$$\beta(y) = \sum_{j \geq 1} \beta_j y^j, \quad \text{with } \beta_1 := \frac{1}{\lambda_1}, \quad \text{and } \beta_2 := \frac{1}{\lambda_1} \left( \frac{1}{2} - \frac{\lambda_2}{\lambda_1^2} \right),$$

implies that

$$(3.23) \quad \beta'(y)\beta(y)^{-n-1} = \lambda_1^n y^{-n-1} e^{-\beta_2 n y / \beta_1} (1 + O(|y| + n|y|^2)),$$

for small  $|y|$ . Consequently, when  $y \asymp n^{-1}$  is small, we get

$$(3.24) \quad \Psi_n(y) = \lambda_1^n [y^n] e^{-\beta_2 n y / \beta_1} (1 + O(|y| + n|y|^2)) \sum_{0 \leq k \leq \lfloor \frac{1}{2}n \rfloor} E_k(y)R_k(y, v).$$

Following the proof of Proposition 9, we know that only a small neighborhood of  $(k, y) \sim (qn, (\mu n)^{-1})$  contributes dominantly; furthermore, the extra factor before the sum in (3.24) is bounded when  $y \asymp n^{-1}$ . Thus we substitute  $y = r$ , with  $r$  expanding as in (3.22), and obtain

$$(3.25) \quad e^{-\beta_2 n y / \beta_1} = e^{-\beta_2 / (\beta_1 \mu)} (1 + O(n^{-\frac{1}{2}})).$$

This, together with (3.24) and (3.8) of Proposition 9, implies that

$$\begin{aligned} P_n(e^{i\theta/\sqrt{n}}) &= \lambda_1^n e^{-\beta_2 / (\beta_1 \mu)} (1 + O(n^{-\frac{1}{2}})) [y^n] E(y, e^{i\theta/\sqrt{n}}) \\ &\quad + O\left(|([y^{n-1}] + n[y^{n-2}]) E(y, e^{i\theta/\sqrt{n}})|\right) \\ &= c\sqrt{n}(\mu\lambda_1)^n n! e^{\mu\sqrt{n}i\theta - \frac{1}{2}\sigma^2\theta^2} (1 + O((1 + |\theta| + |\theta|^3)n^{-\frac{1}{2}})), \end{aligned}$$

where

$$c := \frac{12\sqrt{3}}{\pi^{5/2}} e^{\frac{\pi^2}{6} \left( \frac{\lambda_2}{\lambda_1^2} - \frac{1}{2} \right)}.$$

Now

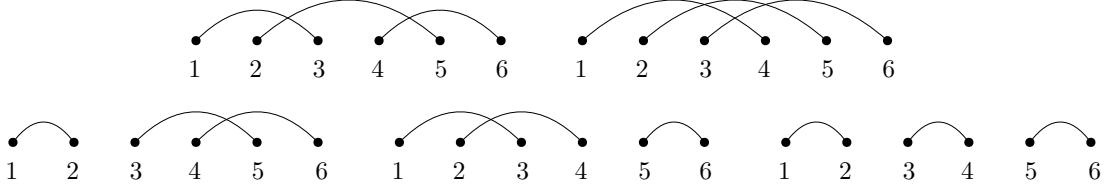
$$\mathbb{E}\left(e^{(X_n - \mu n)i\theta/\sqrt{\sigma^2 n}}\right) = \frac{P_n(e^{i\theta/\sqrt{n}})}{P_n(1)} = e^{\mu\sqrt{n}i\theta - \frac{1}{2}\sigma^2\theta^2} (1 + O((1 + |\theta| + |\theta|^3)n^{-\frac{1}{2}})),$$

implying Theorem 1 by the continuity theorem for characteristic functions.

#### 4. PROOF OF THE STOIMENOW CONJECTURE (THEOREM 4)

We begin by reviewing some necessary definitions on the chord diagrams for our purposes. A *regular linearized chord diagram* (*regular LCD*), also known as a Stoimenow matching, is a matching of the set  $[2n] = \{1, 2, \dots, 2n\}$ , namely, it is a partition of  $[2n]$  into subsets of size exactly two. Each of the subsets is called an *arc*. A matching is a *regular LCD* if it has *no* nested pairs of arcs such that either the openers or the closers are next to each other.

The number of regular linearized chord diagram of size  $n$  (length  $2n$ ) equals the  $n$ -th Fishburn number  $f_n$ ; see for instance [4, 23] and Figure 4.1. By exploiting the relation between the generating



**Figure 4.1.** All regular LCDs of size 3 where the two on top are connected ones.

function (1.1) of the Fishburn numbers and the “half derivative” of the Dedekind eta-function, Zagier [23] derived the asymptotic behavior of Fishburn numbers  $f_n$ , which is our (1.3) with  $\lambda_1 = \lambda_2 = 1$ :

$$(4.1) \quad f_n := [z^n] \sum_{k \geq 0} \prod_{1 \leq j \leq k} (1 - (1 - z)^{-j}) = cn^{\frac{1}{2}} \mu^n n! (1 + O(n^{-1})),$$

where  $(c, \mu) := \left(\frac{12\sqrt{3}}{\pi^{5/2}} e^{\frac{\pi^2}{12}}, \frac{6}{\pi^2}\right)$ .

Given any regular LCD  $D$ , consider a graph  $G_D$  (also known as *intersection graph*) whose vertices are arcs of  $D$  and two vertices are connected by an edge if the corresponding arcs cross each other; in this case if  $G_D$  is connected, then  $D$  is also said to be *connected*; see Figure 4.1 for an illustrative example. The corresponding generating function  $g(z)$  of connected regular LCDs counted by size is given in (1.14).

We now prove Theorem 4, showing that the probability of a uniformly generated large random regular LCD being connected is asymptotic to  $e^{-1}$ .

*Proof.* We are going to prove

$$(4.2) \quad \frac{[z^n]g(z)}{[z^n]\Phi(z, 0)} = \frac{g_n}{f_n} = e^{-1}(1 + O(n^{-1})),$$

where  $[z^n]\Phi(z, 0) = f_n$  is the  $n$ -th Fishburn number. In view of (4.1), it remains to evaluate  $g_n$  as  $n$  tends to infinity.

We first rewrite (1.14) by using the two relations (1.15) and (3.2), giving

$$(4.3) \quad \Phi(z, v) = \frac{v}{(1+v)^2} + \frac{1}{(1+v)^2} \sum_{k \geq 0} \frac{1}{(1-z)^{k+1}} \prod_{1 \leq j \leq k} \frac{((1-z)^{-j} - 1)^2}{1 + v(1-z)^{-j}}.$$

As a result, the equation  $\Phi(z, g(z)) = 1$  can be written as

$$(4.4) \quad g(z)^2 + g(z) + 1 = W(z) := \sum_{k \geq 0} \frac{1}{(1-z)^{k+1}} \prod_{1 \leq j \leq k} \frac{((1-z)^{-j} - 1)^2}{1 + g(z)(1-z)^{-j}}.$$



Taking the coefficients of  $z^n$  on both sides yields

$$(4.5) \quad [z^n]g(z) + [z^n]g(z)^2 = [z^n]W(z), \quad (n \geq 1).$$

While this equation is still recursive, the dependence of the first-order asymptotic approximation on  $g$  is however weak, and indeed only on the coefficient  $g'(0) = 1$  (see (1.13)), similar to the two examples in [14, §6.1.4].

Technically, comparing the right-hand side of (4.4) with the expression (by (1.1) and (2.9))

$$f_n = [z^n] \sum_{k \geq 0} \frac{1}{(1-z)^{k+1}} \prod_{1 \leq j \leq k} ((1-z)^{-j} - 1)^2,$$

we see that the limiting constant  $e^{-1}$  will come from the extra product

$$\bar{W}_k(z) := \prod_{1 \leq j \leq k} \frac{1}{1 + g(z)(1-z)^{-j}}.$$

For the analysis, we first truncate the series to a polynomial:

$$[z^n] \bar{W}_k(z) = [z^n] \prod_{1 \leq j \leq k} \frac{1}{1 + \bar{g}_n(z)(1-z)^{-j}},$$

where  $\bar{g}_n(z) := \sum_{1 \leq \ell \leq n} g_\ell z^\ell$ . Then we prove that  $\bar{g}_n(z) = O(|z|)$  when  $|z| \leq \frac{\varepsilon}{\mu} - \varepsilon$ . By the trivial bound  $g_n \leq f_n$  and the estimate (4.1), we have, when  $|z| = \frac{\varepsilon}{n}$ ,

$$|\bar{g}_n(z)| \leq \bar{g}_n(|z|) = O\left(\sum_{1 \leq \ell \leq n} f_\ell |z|^\ell\right) = O\left(\sum_{1 \leq \ell \leq n} \sqrt{\ell} \mu^\ell \ell! \varrho^\ell n^{-\ell}\right),$$

which is bounded above by  $O(n^{-1})$ :

$$O\left(\sum_{1 \leq \ell \leq n} \ell (\varrho \mu e^{-1} \ell n^{-1})^\ell\right) = O(n^{-1}),$$

whenever  $\varrho < \frac{\varepsilon}{\mu}$ . It follows that, with the same  $|z| = \frac{\varepsilon}{n}$ ,

$$\bar{W}_k(z) = O\left(\exp\left(|z| \sum_{1 \leq j \leq k} (1-|z|)^{-j}\right)\right) = O(1),$$

whenever  $\varrho < \frac{\varepsilon}{\mu}$ . Thus the extra product  $\bar{W}_k(z)$  will only affect the constant term in the asymptotic analysis of  $g_n$ .

Indeed, from the proof of Theorem 1 or Proposition 9, it suffices to examine the behavior of this finite product when  $k = qn + O(n^{\frac{1}{2}+\varepsilon})$  and  $z = (n\mu)^{-1}(1 + O(n^{-\frac{1}{2}}))$ . Note that  $q = \mu \log 2 < \frac{\varepsilon}{\mu}$ . Since  $\bar{g}_n(z) = z + O(|z|^2)$  (when  $z \asymp n^{-1}$ ), we then get, for such  $k$  and  $z$ ,

$$\begin{aligned} \prod_{1 \leq j \leq k} \frac{1}{1 + \bar{g}_n(z)(1-z)^{-j}} &= \exp\left(-\bar{g}_n(z) \sum_{1 \leq j \leq k} (1-z)^{-j} + O\left(|z|^2 \sum_{1 \leq j \leq k} |1-z|^{-2j}\right)\right) \\ &= \exp\left(-(z + O(|z|^2)) \frac{(1-z)^{-k} - 1}{z} + O(|z|)\right) \\ &= e^{-(e^{q/\mu}-1)}(1 + O(n^{-\frac{1}{2}})) = e^{-1}(1 + O(n^{-\frac{1}{2}})). \end{aligned}$$

The detailed proof follows the same procedure we used for the proof of Theorem 1, and is omitted here. We thus obtain

$$(4.6) \quad w_n := [z^n]W(z) = c'n^{\frac{1}{2}}\mu^n n!(1 + O(n^{-1})), \quad \text{with} \quad c' := \frac{12\sqrt{3}}{\pi^{5/2}} e^{\frac{\pi^2}{12}-1}.$$

This gives an approximation of the right-hand-side of (4.5). It remains to prove that  $g_n = [z^n]g(z)$  is asymptotic to  $w_n$ . Note that (4.5) implies that

$$(4.7) \quad \begin{aligned} [z^n]g(z) &= \sum_{1 \leq \ell \leq n} \frac{(-1)^{\ell-1}}{\ell} \binom{2\ell}{\ell} [z^n](W(z) - 1)^\ell \\ &= w_n - \sum_{1 \leq j < n} w_j w_{n-j} + 2 \sum_{\substack{j_1+j_2+j_3=n \\ 1 \leq j_1, j_2, j_3 < n}} w_{j_1} w_{j_2} w_{j_3} - \dots \end{aligned}$$

Here the central binomial coefficients in (4.7) only increase exponentially, while  $w_n$  grows factorially. Now, by (4.6),

$$\begin{aligned} \sum_{1 \leq j < n} w_j w_{n-j} &= 2w_{n-1} + O\left(\sqrt{n} \mu^n \sum_{2 \leq j \leq n-2} j!(n-j)!\right) \\ &= O(n^{-1}w_n + \sqrt{n} \mu^n (n-1)!) \\ &= O(n^{-1}w_n), \end{aligned}$$

because  $j!(n-j)!$  decreases in  $j \in [0, \frac{1}{2}n]$ . Similarly,

$$[z^n](W(z) - 1)^\ell = O(n^{-\ell}w_n), \quad (\ell = 1, 2, \dots),$$

showing that (4.7) is itself an asymptotic expansion. Thus

$$g_n = w_n(1 + O(n^{-1})),$$

which, together with (4.6) and (4.1), proves the limiting ratio (4.2), and thus Theorem 4.  $\square$

Finer approximations for the ratio  $\frac{g_n}{f_n}$  can be derived; for example, we have

$$\frac{g_n}{f_n} = e^{-1} \left( 1 - \frac{\pi^2}{8n} + O(n^{-2}) \right).$$

## 5. SIZE DISTRIBUTION OF RANDOM FMS

We prove in this section Theorem 3, which is an extension of the open problem 5.5 by Jelínek [15]. Unlike the analytic proof for Theorem 1, our approach to Theorem 3 builds on a simple partial fraction decomposition. We also briefly discuss a few other sequences of a similar nature.

**5.1. Asymptotic normality of the size.** We recall that the generating function  $F(z, v)$  of  $\Lambda$ -FMs is given by (3.2). Our study of size distribution is restricted to the situation when  $\Lambda(z)$  is a polynomial. In the special case when  $\Lambda(z) = 1 + z$ , the size of an  $m$ -dimensional primitive FM lies between  $m$  (when only the entries on the main diagonal are 1) and  $\binom{m+1}{2}$  (when all entries are 1). Only near the median size  $\frac{1}{4}m(m+3)$  does the number of primitive FMs of dimension  $m$  reach its peak among all other possible sizes, which is also the case when 0's and 1's are allowed to appear equally likely in each entry (except the diagonal). For instance, in Figure 1.2, most primitive FMs of dimension 3 have size 4 or 5.

*Proof.* Let  $Y_m$  be the size of a random  $m \times m$   $\Lambda$ -FM when all  $\Lambda$ -FMs of dimension  $m$  are equally likely to be selected. The corresponding probability generating function of  $Y_m$  is given by

$$\mathbb{E}(z^{Y_m}) = \frac{[v^m]F(z, v)}{[v^m]F(1, v)},$$

with  $F$  given in (3.2). By partial fraction expansion,

$$\prod_{1 \leq j \leq k} \frac{\Lambda(z)^j - 1}{1 + v(\Lambda(z)^j - 1)} = \sum_{1 \leq j \leq k} \frac{C_{k,j}(z)}{1 + v(\Lambda(z)^j - 1)},$$

where

$$\begin{aligned} C_{k,j}(z) &:= (\Lambda(z)^j - 1)^k \prod_{l \neq j, 1 \leq l \leq k} \frac{\Lambda(z)^l - 1}{\Lambda(z)^j - \Lambda(z)^l} \\ &= (\Lambda(z)^j - 1)^k \left( \prod_{1 \leq l < j} \frac{\Lambda(z)^l - 1}{\Lambda(z)^j - \Lambda(z)^l} \right) \left( \prod_{j < l \leq k} \frac{\Lambda(z)^l - 1}{\Lambda(z)^j - \Lambda(z)^l} \right) \\ &= (-1)^{k-j} (\Lambda(z)^j - 1)^k \Lambda(z)^{-\binom{j}{2}} \prod_{1 \leq l \leq k-j} \frac{1 - \Lambda(z)^{-j-l}}{1 - \Lambda(z)^{-l}}. \end{aligned}$$

Consequently,

$$\begin{aligned} [v^m]F(z, v) &= \sum_{1 \leq k \leq m} \sum_{1 \leq j \leq k} C_{k,j}(z) (-1)^{m-k} (\Lambda(z)^j - 1)^{m-k} \\ &= \sum_{1 \leq k \leq m} \sum_{1 \leq j \leq k} (-1)^{m-j} (\Lambda(z)^j - 1)^m \Lambda(z)^{-\binom{j}{2}} \left( \prod_{1 \leq l \leq k-j} \frac{1 - \Lambda(z)^{-l-j}}{1 - \Lambda(z)^{-l}} \right). \end{aligned}$$

Interchanging the two sums and rearranging the sum-indices lead to

$$[v^m]F(z, v) = \Lambda(z)^{\binom{m+1}{2}} \sum_{0 \leq j < m} (-1)^j (1 - \Lambda(z)^{-m+j})^m \Lambda(z)^{-\binom{j+1}{2}} \sum_{0 \leq k \leq j} \prod_{1 \leq l \leq k} \frac{1 - \Lambda(z)^{-l-m+j}}{1 - \Lambda(z)^{-l}}.$$

For our limit law purposes, we consider  $z \sim 1$ . Since  $\Lambda(z)$  is a polynomial with positive coefficients and  $\Lambda(1) > 1$ , there is a small neighborhood of unity, say  $|z - 1| \leq \delta$ ,  $\delta > 0$ , where  $|\Lambda(z)| > 1$ . Thus for such  $z$

$$(1 - \Lambda(z)^{-m+j})^m - 1 = O(m|\Lambda(z)|^{-m+j}).$$

We then deduce that

$$(5.1) \quad [v^m]F(z, v) = H(z) \Lambda(z)^{\binom{m+1}{2}} (1 + O(m|\Lambda(z)|^{-m})),$$

uniformly for  $|z - 1| \leq \delta$ , where

$$H(z) := \sum_{j \geq 0} (-1)^j \Lambda(z)^{-\binom{j+1}{2}} \sum_{0 \leq k \leq j} \prod_{1 \leq l \leq k} \frac{1}{1 - \Lambda(z)^{-l}}.$$

In particular, the total number of  $m$ -dimensional  $\Lambda$ -FMs satisfies

$$(5.2) \quad \frac{[v^m]F(1, v)}{\Lambda(1)^{\binom{m+1}{2}}} \rightarrow H(1) = \sum_{j \geq 0} (-1)^j \Lambda(1)^{-\binom{j+1}{2}} \sum_{0 \leq k \leq j} \frac{1}{Q_k},$$

where

$$Q_k := \prod_{1 \leq \ell \leq k} \frac{1}{1 - \Lambda(1)^{-\ell}}.$$

An alternative expression of  $H(1)$  with additional numerical advantages is

$$H(1) = \frac{1}{Q_\infty} \sum_{j \geq 0} (-1)^j \Lambda(1)^{-(j+1)} \sum_{l \geq 0} \frac{(-1)^j \Lambda(1)^{-\binom{l}{2}}}{Q_l} \cdot \frac{1 - \Lambda(1)^{-l(j+1)}}{1 - \Lambda(1)^{-l}},$$

which follows from the Euler identity

$$\frac{Q_\infty}{Q_k} = \sum_{l \geq 0} \frac{(-1)^l \Lambda(1)^{-\binom{l}{2}}}{Q_l} \Lambda(1)^{-kl}.$$

Furthermore, it also follows from (5.1) that

$$(5.3) \quad \mathbb{E}(z^{Y_m}) = \frac{[v^m]F(z, v)}{[v^m]F(1, v)} = \frac{H(z)}{H(1)} \left( \frac{\Lambda(z)}{\Lambda(1)} \right)^{\binom{m+1}{2}} (1 + O(m\Lambda(1)^{-m} + m|\Lambda(z)|^{-m})),$$

uniformly for  $|z - 1| \leq \delta$ . By applying the Quasi-powers Theorem ([11, IX.5] or [13]), we conclude that the distribution of the random variable  $Y_m$  is asymptotically normally distributed with mean and variance asymptotic to

$$\begin{aligned} \mathbb{E}(Y_m) &= \hat{\mu}m(m+1) + \frac{H'(1)}{H(1)} + O(m\Lambda(1)^{-m}), \\ \mathbb{V}(Y_m) &= \hat{\sigma}^2m(m+1) + \frac{H'(1) + H''(1)}{H(1)} - \left( \frac{H'(1)}{H(1)} \right)^2 + O(m\Lambda(1)^{-m}), \end{aligned}$$

where  $(\hat{\mu}, \hat{\sigma}^2)$  are defined in (1.11). □

As a special case, consider  $\Lambda = \{0, 1, \dots, h-1\}$ , where  $h \geq 2$ . Then  $\Lambda(1) = h$ . The asymptotic expression (5.2) then suggests the following algorithm for generating a random  $m$ -dimensional  $\Lambda$ -FM: Generate first the two corners on the diagonal by two independent integer-valued uniform distribution  $\text{Uniform}[1, h-1]$ , and then the remaining entries  $\text{Uniform}[0, h-1]$ . Reject the matrix if it fails to be Fishburn and stop if it is. The probability of success is given by, according to (5.2),

$$p_h := \frac{h^2}{(h-1)^2} \sum_{j \geq 0} (-1)^j h^{-\binom{j+1}{2}} \sum_{0 \leq k \leq j} \prod_{1 \leq l \leq k} \frac{1}{1 - h^{-l}}.$$

This probability tends to 1 as  $h$  increases.

$h$	2	3	4	5	6	7	8	9	10
$p_h$	0.334	0.706	0.843	0.903	0.935	0.953	0.965	0.972	0.978

The more than doubled jump of the success probability from  $p_2$  to  $p_3$  is more significant than expected; these values also show that the naive rejection method is generally very efficient.

Three sequences are found in the OEIS of the form  $[v^m]F(1, v)$ , and they are summarized in the following table; their asymptotic behaviors are described by (5.2).

OEIS	[20, A005321]	[20, A289314]	[20, A289315]
$\Lambda(z)$	$1 + z$	$1 + z + z^2$	$1 + z + z^2 + z^3$
Generating function $F(1, z)$	$\sum_{k \geq 0} \prod_{1 \leq j \leq k} \frac{(2^j - 1)z}{1 + (2^j - 1)z}$	$\sum_{k \geq 0} \prod_{1 \leq j \leq k} \frac{(3^j - 1)z}{1 + (3^j - 1)z}$	$\sum_{k \geq 0} \prod_{1 \leq j \leq k} \frac{(4^j - 1)z}{1 + (4^j - 1)z}$

**5.2. Andresen and Kjeldsen's 1976 paper.** In a somewhat disguised context of transitively directed graphs, Andresen and Kjeldsen studied in their pioneering paper [1] three sequences connected to primitive FMs, denoted by  $\xi_{m,k}$ ,  $\eta_{m,k}$  and  $\psi_{m,k}$ , respectively. (We change their notation  $f(m, k)$  to  $f_{m,k}$  to reduce the occurrences of parentheses.) We show in this subsection that these three sequences are all asymptotically normally distributed for large  $m$  with mean and variance asymptotic to  $\frac{1}{2}m$  and  $\frac{1}{4}m$ , respectively.

In terms of the matrix language,  $\xi_{m,k}$  counts the number of primitive FMs of dimension  $m$  with first row sum  $k$ , and  $\psi_{m,k}$  the number of upper triangular binary matrices (matrices with entries 0 or 1) of dimension  $m$  with first row sum  $k$  such that

- the  $j$ -th column ( $1 \leq j < m$ ) is a zero column if and only if the  $(j + 1)$ -st row is a zero row;
- all nonzero columns and rows form a primitive FM.

While a combinatorial interpretation of the last sequence  $\eta_{m,k}$  is still lacking (an open question in [1]), its generating polynomial satisfies a similar type of recurrence as that of  $\xi$  and  $\psi$ :

$$\begin{cases} P_m^{[\xi]}(v) = vP_{m-1}^{[\xi]}(1+2v) - vP_{m-1}^{[\xi]}(v), \\ P_m^{[\eta]}(v) = vP_{m-1}^{[\eta]}(1+2v) - (1+v)P_{m-1}^{[\eta]}(v), \\ P_m^{[\psi]}(v) = vP_{m-1}^{[\psi]}(1+2v) + (1-v)P_{m-1}^{[\psi]}(v), \end{cases}$$

for  $n \geq 2$ , all with the same initial conditions  $P_1^{[\cdot]}(v) = v$ . Other types of recurrences are also derived in [1]. These recurrences are readily solved by iterating the corresponding functional equations satisfied by the bivariate generating functions, and we obtain

Sequence	$\xi_{m,k}$	$\eta_{m,k}$	$\psi_{m,k}$
OEIS	[20, A259971]	[20, A259972]	[20, A259970]
Bivariate GF	$\sum_{k \geq 0} \prod_{0 \leq j \leq k} \frac{(2^j - 1 + 2^j v)z}{1 - z + 2^j(1+v)z}$	$\sum_{k \geq 0} \prod_{0 \leq j \leq k} \frac{(2^j - 1 + 2^j v)z}{1 + 2^j(1+v)z}$	$\sum_{k \geq 0} \prod_{0 \leq j \leq k} \frac{(2^j - 1 + 2^j v)z}{1 - 2z + 2^j(1+v)z}$
$\sum_k f_{m,k}$	[20, A005321]	[20, A005014]	[20, A005016]

In particular, by a direct partial fraction expansion,

$$P_m^{[\eta]}(v) = v \sum_{0 \leq k < m} (-1)^k (1+v)^{m-1-k} \prod_{k < l < m} (2^l - 1) \quad (m \geq 1),$$

and the reason of introducing  $\eta_{m,k}$  is because of the relations

$$P_m^{[\xi]}(v) = \sum_{0 \leq j < m} \binom{m-1}{j} P_{m-j}^{[\eta]}(v), \quad \text{and} \quad P_n^{[\psi]}(v) = \sum_{0 \leq j < m} \binom{m-1}{j} 2^j P_{m-j}^{[\eta]}(v).$$

From these forms, the limiting normal distribution in all cases can be derived by a similar argument used above for the size distribution  $Y_m$ . Now write  $Q_m := \prod_{1 \leq j \leq m} (1 - 2^{-j})$ . Then

$$\begin{aligned} P_m^{[\eta]}(v) &= 2^{\binom{m}{2}} \sum_{0 \leq k < m} \frac{(-1)^k 2^{-\binom{k}{2}} Q_{m-1}}{Q_k} (1+v)^{m-1-k} \\ &= T(v) v (1+v)^{m-1} 2^{\binom{m}{2}} \left( 1 + O\left( 2^{\binom{m}{2}} \sum_{k \geq m} 2^{-\binom{k}{2}} |1+v|^{-k} \right) \right), \end{aligned}$$

where

$$T(v) := Q_\infty \sum_{k \geq 0} \frac{(-1)^k 2^{-\binom{k}{2}}}{Q_k} (1+v)^{-k},$$

is a meromorphic function of  $v$ . This implies that

$$\frac{P_m^{[\eta]}(v)}{P_m^{[\eta]}(1)} = \frac{vT(v)}{T(1)} \left( \frac{1+v}{2} \right)^{m-1} (1 + O(|1+v|^{-m})),$$

uniformly for  $|v+1| \geq 1 + \varepsilon$ , and from this we then deduce the asymptotic normality  $\mathcal{N}(\frac{1}{2}m, \frac{1}{4}m)$  for the underlying random variables by Quasi-powers Theorem ([11, IX.5] or [13]). Exactly the same type of results hold for the other two sequences.

**5.3. Some related OEIS sequences.** A few other sequences in the OEIS are closely connected to the sequences we discussed in this section. We list them in the following table. Asymptotic or distributional properties can be dealt with by the same techniques, and are omitted here.

[20, A005327]	$\frac{1}{1+z} \sum_{k \geq 0} \prod_{1 \leq j \leq k} \frac{(2^j-1)z}{1+2^j z}$	[20, A002820]	$2^{\binom{n}{2}} \times \text{A005327}(n+1)$
[20, A005016]	$\sum_{k \geq 0} \prod_{1 \leq j \leq k} \frac{(2^j-1)z}{1+(2^j-2)z}$	[20, A005331]	$\frac{1}{1-z} \sum_{k \geq 0} \prod_{1 \leq j \leq k} \frac{(2^j-1)z}{1+(2^j-2)z}$
[20, A005329]	$\sum_{k \geq 0} \prod_{1 \leq j \leq k} \frac{2^j z}{1+2^j z}$	[20, A028362]	$\sum_{k \geq 0} \prod_{0 \leq j < k} \frac{2^j z}{1-2^j z}$
[20, A182507]	$\sum_{k \geq 0} \prod_{1 \leq j \leq k} \frac{j2^{j-1} z}{1+j2^j z}$	[20, A006116]:	$\sum_{k \geq 0} z^k \prod_{0 \leq j \leq k} \frac{1}{1-2^j z}$

**Remark 3.** *It is interesting to compare [20, A028362] with the two identities*

$$1+z = \sum_{k \geq 0} \prod_{1 \leq j \leq k} \frac{2^{j-1} z}{1+2^j z}, \quad \text{and} \quad \frac{1}{1-z} = \sum_{k \geq 0} \prod_{1 \leq j \leq k} \frac{jz}{1+jz}.$$

*Another sequence with a similar nature is the Gaussian polynomials [20, A022166]:*

$$\sum_{k \geq 0} \prod_{1 \leq j \leq k} \frac{vz}{1-2^j z}.$$

## 6. SELF-DUAL FMS AND FMS WITHOUT 1'S

In this section, we briefly describe the limiting behaviors of random self-dual FMs and random FMs whose smallest nonzero entries are 2, respectively. The asymptotics in both cases are similar and involve a stretched exponential factor of the form  $e^{\Theta(\sqrt{n})}$ . We only sketch the proof in the self-dual case, and omit that in the other.

**6.1. Dimension of self-dual FMs.** The dimension distribution of  $\Lambda$ -FMs (Theorem 1) exhibits a limiting invariance property in the sense that the central limit theorem is independent of the entry-set  $\Lambda$  as long as  $\lambda_1 > 0$ . We show here that the same limiting property holds even when we restrict our random matrices to be self-dual (or persymmetric). What is less expected here is that the variance in the random self-dual FMs is asymptotically double that in the ordinary case (while the mean remains asymptotically the same); see Table 3 for a numerical illustration in the case of primitive FMs (with  $\Lambda(z) = 1+z$ ).

Self-dual primitive FMs							Primitive FMs									
$n \setminus k$	1	2	3	4	5	6	$(\mu_n, \sigma_n^2)$	$n \setminus k$	1	2	3	4	5	6	$(\mu_n, \sigma_n^2)$	
1	1						(1, 0)	1	1						(1, 0)	
2		1					(2, 0)	2		1					(2, 0)	
3			1	1			$(\frac{5}{2}, \frac{1}{4})$	3			1	1			$(\frac{5}{2}, \frac{1}{4})$	
4				2	1		$(\frac{10}{3}, \frac{2}{9})$	4			4	1			$(\frac{16}{5}, \frac{4}{25})$	
5					2	3	1	$(\frac{23}{6}, \frac{17}{36})$	5			4	11	1	$(\frac{61}{16}, \frac{71}{256})$	
6						1	5	6	1			1	33	26	1	$(\frac{271}{61}, \frac{1162}{3721})$

**Table 3.** The first few values of the dimension statistics in self-dual primitive FMs and ordinary primitive FMs (where  $n \setminus k = \text{size} \setminus \text{dimension}$ ). In particular, among the 5 primitive FMs of size 4, only 3 are self-dual, resulting in higher variance, and a similar observation holds for matrices of larger size.

**Theorem 15.** Let  $Z_n$  denote the dimension of a random self-dual  $\Lambda$ -FM, where all size- $n$  self-dual  $\Lambda$ -FMs are equally likely. If  $\Lambda(z)$  is analytic at  $z = 0$  with  $\lambda_1 > 0$ , then  $Z_n$  is asymptotically normally:

$$\frac{Z_n - \mu n - \mu' \sqrt{n}}{\sigma \sqrt{2n}} \xrightarrow{d} \mathcal{N}(0, 1),$$

where  $(\mu, \sigma)$  is defined in (1.6),

$$\mu' := \frac{\sqrt{6}}{2\pi^3} (12 \log 2 - \pi^2 \sqrt{\lambda_1}),$$

and the mean and the variance are asymptotic to

$$(6.1) \quad \begin{aligned} \mathbb{E}(Z_n) &= \mu n + \mu' \sqrt{n} + O(1), \\ \mathbb{V}(Z_n) &= 2\sigma^2 n + \frac{\sqrt{6}}{4\pi^5} (24(18 - \pi^2) \log 2 - \pi^2(24 - \pi^2) \sqrt{\lambda_1}) \sqrt{n} + O(1), \end{aligned}$$

respectively.

*Proof.* Our analysis is based on the generating function  $G(z, v)$  for the dimension (marked by  $v$ ) of self-dual primitive FMs of a given size (as marked by  $z$ ) derived by Jelínek in [15]:

$$(6.2) \quad G(z, v) + v = \sum_{k \geq 1} \Lambda(z)^k \Lambda(z^2)^{\binom{k}{2}} v^{2k-1} \frac{1 + v(\Lambda(z^2)^k - 1)}{\Lambda(z^2)^k - 1} \prod_{1 \leq j \leq k} \frac{\Lambda(z^2)^j - 1}{1 + v^2(\Lambda(z^2)^j - 1)}.$$

When  $v = 1$ , we have

$$G(z, 1) + 1 = \sum_{k \geq 1} G_k(z), \quad \text{with} \quad G_k(z) := \Lambda(z)^k \prod_{1 \leq j < k} (\Lambda(z^2)^j - 1).$$

Asymptotic approximation of  $[z^n]G(z, 1)$  was already derived in [14]; in particular, when  $\lambda_1 > 0$ ,

$$[z^n]G(z, 1) = ce^{\beta \sqrt{n}} (\lambda_1 \mu)^{\frac{1}{2}n} n^{\frac{1}{2}(n+1)} (1 + O(n^{-\frac{1}{2}})),$$

where

$$(c, \beta, \mu) := \left( \frac{3\sqrt{2}}{\pi^{3/2}} 2^{\frac{\lambda_2}{2} - \frac{\lambda_1}{2}} e^{-\frac{\lambda_1}{4} - \frac{\pi^2}{24} + \frac{\pi^2 \lambda_2}{12\lambda_1^2} + \frac{3\lambda_1}{2\pi^2} (\log 2)^2}, \frac{\sqrt{6\lambda_1}}{\pi} \log 2, \frac{6}{e\pi^2} \right).$$

A finer expansion can be derived, which is of the form

$$(6.3) \quad [z^n] \sum_{k \geq 0} \Lambda(z)^k \prod_{1 \leq j < k} (\Lambda(z^2)^j - 1) = ce^{\beta\sqrt{n}} \mu^{\frac{1}{2}n} n^{\frac{1}{2}n+1} \left( 1 + \sum_{j \geq 1} \bar{d}_j n^{-\frac{1}{2}j} \right),$$

for some computable coefficients  $\bar{d}_j$ . We compute first the mean. For convenience, write  $\Lambda_j := \Lambda(z^j)$ . Taking the derivative with respect to  $v$  and substituting  $v = 1$  on both sides of (6.2) give

$$\begin{aligned} M_1(z) &:= \partial_v G(z, v)|_{v=1} = \sum_{k \geq 1} G_k(z) \left( \frac{2}{\Lambda_2 - 1} - \frac{\Lambda_2^{-k} (\Lambda_2 + 1)}{\Lambda_2 - 1} \right) - 1 \\ &= \frac{2(G(z, 1) + 1)}{\Lambda_2 - 1} - \frac{\Lambda_2 + 1}{\Lambda_2 - 1} \sum_{k \geq 1} \frac{\Lambda_1^k}{\Lambda_2^k} \prod_{1 \leq j < k} (\Lambda_2^j - 1) - 1. \end{aligned}$$

Let now

$$\begin{aligned} S_1(z) &:= \sum_{k \geq 1} \frac{\Lambda_1^k}{\Lambda_2^k} \prod_{1 \leq j < k} (\Lambda_2^j - 1) = \sum_{k \geq 0} \frac{\Lambda_1^{k+1}}{\Lambda_2^{k+1}} \prod_{1 \leq j \leq k} (\Lambda_2^j - 1) \\ &= \frac{\Lambda_1}{\Lambda_2} + \frac{\Lambda_1}{\Lambda_2} \sum_{k \geq 1} \frac{\Lambda_1^k (\Lambda_2^k - 1)}{\Lambda_2^k} \prod_{1 \leq j < k} (\Lambda_2^j - 1) \\ &= \frac{\Lambda_1}{\Lambda_2} (G(z, 1) + 2) - \frac{\Lambda_1}{\Lambda_2} S_1(z). \end{aligned}$$

Thus

$$S_1(z) = \frac{\Lambda_1}{\Lambda_1 + \Lambda_2} (G(z, 1) + 2),$$

and, consequently,

$$M_1(z) = \frac{2\Lambda_2 - \Lambda_1(\Lambda_2 - 1)}{(\Lambda_2 - 1)(\Lambda_1 + \Lambda_2)} (G(z, 1) + 2) - \frac{2}{\Lambda_2 - 1} - 1.$$

By (6.3), we then deduce the asymptotic approximation of the mean, as that given in (6.1).

For the variance, we compute first the second moment. By the same argument, we have

$$\begin{aligned} M_2(z) &:= \partial_v^2 G(z, v)|_{v=1} + \partial_v G(z, v)|_{v=1} \\ &= \sum_{k \geq 1} G_k(z) \left( \frac{4(\Lambda_2^2 + 1)\Lambda_2^{-2k}}{(\Lambda_2 - 1)(\Lambda_2^2 - 1)} + \frac{(\Lambda_2^2 - 2\Lambda_2 - 7)\Lambda_2^{-k}}{(\Lambda_2 - 1)^2} - \frac{4(\Lambda_2^2 - 2\Lambda_2 - 1)}{(\Lambda_2 - 1)(\Lambda_2^2 - 1)} \right) - 1. \end{aligned}$$

Let

$$S_2(z) := \sum_{k \geq 1} \frac{\Lambda_1^k}{\Lambda_2^{2k}} \prod_{1 \leq j < k} (\Lambda_2^j - 1).$$



Then

$$\begin{aligned} S_2(z) &= \frac{\Lambda_1}{\Lambda_2^2} + \frac{\Lambda_1^2(\Lambda_2 - 1)}{\Lambda_2^4} + \frac{\Lambda_1^2}{\Lambda_2^3} \sum_{k \geq 1} \Lambda_1^k \left( 1 - \frac{\Lambda_2 + 1}{\Lambda_2^{k+1}} + \frac{1}{\Lambda_2^{2k+1}} \right) \prod_{1 \leq j < k} (\Lambda_2^j - 1) \\ &= \frac{\Lambda_1}{\Lambda_2^2} + \frac{\Lambda_1^2(\Lambda_2 - 1)}{\Lambda_2^4} + \frac{\Lambda_1^2}{\Lambda_2^3} \left( G(z, 1) + 1 - \frac{\Lambda_2 + 1}{\Lambda_2} S_1(z) + \frac{S_2(z)}{\Lambda_2} \right), \end{aligned}$$

which is solved to be

$$S_2(z) = \frac{\Lambda_1^2}{(\Lambda_1 + \Lambda_2)(\Lambda_1 + \Lambda_2^2)} (G(z, 1) + 2) + \frac{\Lambda_1}{\Lambda_1 + \Lambda_2^2}.$$

It follows that

$$\begin{aligned} M_2(z) &= \left( \frac{\Lambda_1}{\Lambda_1 + \Lambda_2} + \frac{4\Lambda_2}{(\Lambda_2 - 1)^2(\Lambda_2^2 - 1)} \left( \frac{3\Lambda_2^2 - 1}{\Lambda_1 + \Lambda_2} - \frac{\Lambda_2^2(\Lambda_2^2 + 1)}{\Lambda_1 + \Lambda_2^2} \right) \right) (G(z, 1) + 2) \\ &\quad + \frac{8\Lambda_2}{\Lambda_2^2 - 1} - \frac{4\Lambda_2^2(\Lambda_2^2 + 1)}{(\Lambda_2 - 1)(\Lambda_2^2 - 1)(\Lambda_1 + \Lambda_2^2)} - 1. \end{aligned}$$

From this expression and the expansion (6.3), we deduce an asymptotic approximation to the second moment  $\mathbb{E}(Z_n^2)$ , and then the asymptotic variance in (6.1).

The proof for the normal limit law is similar to that of Theorem 1, with the modifications needed to incorporate the change at the order  $\sqrt{n}$ . We list here the major steps. Prove first that when  $\Lambda(z) = e^z$  and  $v = e^{\theta/\sqrt{n}}$ ,

$$[z^n]G(z, v) = ce^{\beta\sqrt{n}} \rho^{\frac{1}{2}n} n^{\frac{1}{2}(n+1)} e^{(\mu\sqrt{n} + \mu')\theta + \sigma^2\theta^2} (1 + O(n^{-\frac{1}{2}})),$$

where

$$(c, \beta, \rho) = \left( \frac{3\sqrt{2}}{\pi^{3/2}} e^{-\frac{1}{4} + \frac{3}{2\pi^2}(\log 2)^2}, \sqrt{\mu} \log 2, \frac{\mu}{e} \right).$$

Then, by the change of variables  $\Lambda(z^2) = e^{y^2}$ , and by following the same analysis, we deduce that

$$[z^n]G(z, v) = ce^{\beta\sqrt{n}} \rho^{\frac{1}{2}n} n^{\frac{1}{2}(n+1)} e^{(\mu\sqrt{n} + \mu')\theta + \sigma^2\theta^2} (1 + O((|\theta| + |\theta|^3)n^{-\frac{1}{2}})),$$

uniformly for  $|\theta| = o(n^{-1/6})$ , where

$$(c, \beta, \rho) = \left( \frac{3\sqrt{2}}{\pi^{3/2}} 2^{\frac{\lambda_2}{\lambda_1} - \frac{\lambda_1}{2}} e^{-\frac{\lambda_1}{4} - \frac{\pi^2}{24} + \frac{\pi^2\lambda_2}{12\lambda_1^2} + \frac{3\lambda_1}{2\pi^2}(\log 2)^2}, \sqrt{\mu\lambda_1} \log 2, \frac{\mu\lambda_1}{e} \right).$$

This proves the asymptotic normality of  $Z_n$ .  $\square$

**6.2. FMs without 1's.** What happens if  $\lambda_1 = 0$  and the smallest nonzero entry is 2? In this case, the generating functions remain the same but with  $\Lambda(z) = 1 + \lambda_2 z^2 + \dots$ . Following the asymptotic approximations derived in [14], we can also prove the corresponding central limit theorem for the dimension of random FMs.

**Theorem 16.** *Assume that  $\Lambda(z)$  is analytic at  $z = 0$  with  $\lambda_1 = 0$ ,  $\lambda_2 > 0$  and that all such  $\Lambda$ -FMs of size  $n$  are equally likely to be selected. Then the dimension  $X_n$  of a random matrix is asymptotically normally distributed with mean and variance both linear in  $n$ :*

$$(6.4) \quad \frac{X_n - \bar{\mu}n - \bar{\mu}'\sqrt{n}}{\bar{\sigma}\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, 1), \quad \text{with} \quad (\bar{\mu}, \bar{\mu}', \bar{\sigma}^2) := \left( \frac{3}{\pi^2}, -\frac{\sqrt{3}\lambda_3}{2\pi\lambda_2^{3/2}}, \frac{3(12 - \pi^2)}{2\pi^4} \right),$$

so that  $\bar{\mu} = \frac{1}{2}\mu$  and  $\bar{\sigma}^2 = \frac{1}{2}\sigma^2$ , where  $(\mu, \sigma^2)$  is given in (1.6), and

$$\begin{aligned}\mathbb{E}(X_n) &= \bar{\mu}n + \bar{\mu}'\sqrt{n} + O(1), \\ \mathbb{V}(X_n) &= \bar{\sigma}^2n + \left( \frac{\lambda_3(\pi^2 - 6)}{4\pi^2\lambda_2^{3/2}} - \frac{2\lambda_5}{\lambda_2^{5/2}} + \frac{4\lambda_3\lambda_4}{\lambda_2^{7/2}} - \frac{2\lambda_3^3}{\lambda_2^{9/2}} \right) \sqrt{\bar{\mu}n} + O(1),\end{aligned}$$

respectively.

Both the mean and the variance constants are halved, when compared with those of the  $\lambda_1 > 0$  case. This is intuitively clear as one expects that the entry 2 is omnipresent.

The analysis of this well anticipated limit result is much more involved than it looks because the polynomial term in the asymptotic approximation depends on the first nonzero odd number in the entry-set  $\Lambda$ . More precisely, if

$$\lambda_{2j-1} = 0, \quad \text{for } 1 \leq j \leq \ell, \quad \text{and } \lambda_2, \lambda_{2\ell+1} > 0,$$

then it is proved in [14] that the total number of  $\Lambda$ -FMs of size  $n$  satisfies

$$[z^n] \sum_{k \geq 0} \prod_{1 \leq j \leq k} (1 - \Lambda(z)^{-j}) = c_\ell e^{\beta\sqrt{n}} \rho^{\frac{1}{2}n} n^{\frac{1}{2}n + \chi_\ell} (1 + O(n^{-\frac{1}{2}})),$$

where  $(c_\ell, \chi_\ell)$  depends not only on  $\ell$  but also on the parity of  $n$ , and

$$(\beta, \rho) = \left( \frac{\lambda_3\pi}{2\sqrt{3}\lambda_2^{3/2}}, \frac{3\lambda_2}{e\pi^2} \right).$$

## 7. CONCLUDING REMARKS

We conclude this paper by briefly indicating possible refinements to the central limit theorems derived in this paper.

The simplest case is Theorem 3 concerning the size distribution of random FMs under fixed dimensions. Since it fits the standard Quasi-powers framework by (5.3), an optimal rate of order  $O(m^{-1})$  is readily guaranteed; see [13].

Optimal convergence rates in the other cases are structurally well expected, but technically more involved. For instance, a closer examination of our proof of Proposition 9 shows that the proof given there holds indeed in the wider range  $|v - 1| \leq \varepsilon$ ; that is, we have the more precise estimate

$$\mathbb{E}(e^{X_n i\theta/\sqrt{n}}) = \begin{cases} e^{\mu\sqrt{n}i\theta - \frac{1}{2}\sigma^2\theta^2} (1 + O((|\theta| + |\theta|^3)n^{-\frac{1}{2}})), & \text{if } |\theta| \leq \varepsilon n^{\frac{1}{6}}, \\ O(e^{-\varepsilon\theta^2}), & \text{if } \varepsilon n^{\frac{1}{6}} \leq |\theta| \leq \varepsilon n^{\frac{1}{2}}. \end{cases}$$

Indeed, our proof of Proposition 9 is simpler if we restrict to the range  $|v - 1| = O(n^{-\frac{1}{2}})$  for central limit theorem purposes. By the classical Berry-Esseen inequality (see [11, p. 641] or [13] and the references therein), we can then obtain an optimal convergence rate of order  $O(n^{-\frac{1}{2}})$  in the central limit theorem (1.6), namely,

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( \frac{X_n - \mu n}{\sigma\sqrt{n}} \leq x \right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}t^2} dt \right| = O(n^{-\frac{1}{2}}).$$

Similarly, optimal convergence rates can be derived for other central limit theorems, namely, Theorem 15 and Theorem 16.

Finally, local limit theorems are also anticipated, but the technicalities involved are more delicate; these and related approximations will be discussed elsewhere.

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