Asymptotics and statistics on Fishburn matrices and their generalizations

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Abstract. A direct saddle-point analysis (without relying on any modular forms or functional equations) is developed to establish the asymptotics of Fishburn matrices and a large number of other variants with a similar sum-of-finite-product form for their (formal) generating functions. In addition to solving some conjectures, the application of our saddle-point approach to the distributional aspects of statistics on Fishburn matrices is also examined with many new limit theorems characterized, representing the first of their kind for such structures.

Keywords: Fishburn matrices; Fishburn numbers; saddle-point method; q-series; generating functions; Stirling statistics.

1. Motivations and Background

Fishburn matrices, introduced in the 1970s in the context of interval orders (in order theory) and directed graphs (see [1, 18, 23, 24]), are upper-triangular matrices with nonnegative integers as entries such that no row and no column contains exclusively zeros. They have been later found to be bijectively equivalent to several other combinatorial structures such as $(\mathbf{2}+\mathbf{2})$-free posets, ascent sequences, certain pattern-avoiding permutations, $(\mathbf{2}-\mathbf{1})$-avoiding inversion sequences, Stoimenow matchings, and regular linearized chord diagrams; see, for instance, [6, 14, 21, 31, 36] and Section 2 for more information.

In addition to their rich combinatorial connections and modeling capabilities, the corresponding asymptotic enumeration and the finer distributional properties are equally enriching and challenging, as we will explore in this paper. In particular, while the asymptotics of some classes of Fishburn matrices were known (see, for example, [8, 45]), the stochastic aspects of the major characteristic statistics on random Fishburn matrices have remained open up to now.

Zagier, in his influential paper [45] on Vassiliev invariants and quantum modular forms, derived the asymptotic approximation to the number of Fishburn matrices whose entries sum to n

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(1-(1-z)^j\right) = c\rho^n\, n^{n+1}\left(1+O\big(n^{-1}\big)\right), \tag{1.1}$$

(see OEIS [37] sequence A022493, the Fishburn numbers), where $(c,\rho) := \big(\frac{12\sqrt{6}}{\pi^2}\, e^{\frac{\pi^2}{12}}, \frac{6}{e\pi^2}\big)$. Here $[z^n]f(z)$ denotes the coefficient of z^n in the (formal) Taylor expansion of f and all A-numbers (followed by six digits) refer to sequences from the OEIS [37]. For conciseness of

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notation and readability, *all constant pairs* (c, ρ) *throughout this paper are generic and may not be the same at each occurrence*; their values will be locally specified.

That the asymptotic approximation (1.1) is remarkable can be viewed in various perspectives. First, the Taylor coefficients of the inner product on the left-hand side of (1.1) alternate in sign, so it is unclear if the coefficient of z^n in the sum-of-product expression is positive for all positive n, much less its large factorial growth order shown on the right-hand side. Second, since $\frac{6}{\pi^2} < 1$, the right-hand side of (1.1) is exponentially smaller than $n!$, which equals $[z^n]\prod_{1\leqslant j\leqslant n}\big(1-(1-z)^j\big)$. More precisely, we will prove that (see Lemma 11 and Proposition 12)

$$\max_{1\leqslant k\leqslant n}\Big|[z^n]\prod_{1\leqslant j\leqslant k}\big(1-(1-z)^j\big)\Big| = \max_{1\leqslant k\leqslant n}[z^n]\prod_{1\leqslant j\leqslant k}\big((1+z)^j-1\big) = \Theta\big(n^{n+\frac{1}{2}}\hat{\rho}^n\big),$$

meaning that the left-hand side is exactly of order $n^{n+\frac{1}{2}}\hat{\rho}^n$ with $\hat{\rho} := \frac{12}{e\pi^2} = 2\rho$. This shows that there is indeed a heavy *exponential cancellation* involved in the sum on the left-hand side of (1.1). Third, in addition to the connection to linearly independent Vassiliev invariants, the Fishburn numbers have now been known to enumerate many different combinatorial objects; see Section 2, OEIS sequence A022493 and [4, 6, 8, 9, 13, 14, 16, 18, 31, 32, 34, 36, 44, 45] for more information. Finally, Zagier's proof of (1.1) relies crucially on an unusual pair of identities

$$\begin{cases} e^{-\frac{z}{24}}\displaystyle\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(1-e^{-jz}\big) = \sum_{n\geqslant 0}\frac{T_n}{n!}\left(\frac{z}{24}\right)^n, \\ \displaystyle\sum_{n\geqslant 0}\frac{T_n}{(2n+1)!}\,z^{2n+1} := \frac{\sin(2z)}{2\cos(3z)}, \end{cases} \tag{1.2}$$

where the integers T_n's, known as Glaisher's T-numbers (see A002439), are defined by the second identity of (1.2). The first one, due to Zagier, is a consequence of the relation between the generating function in (1.1) and the "half derivative" of the Dedekind eta-function, partial summation, Euler's pentagonal number theorem, functional equations, Dirichlet series and Mellin transform techniques; see [45, 46]. The asymptotics of T_n is then readily computed by, say using the singularity analysis (see [20]) on the right-hand side of the second identity, which, unlike the formal nature of the first, is analytic in $|z| < \frac{\pi}{6}$. What appears to be more important in subsequent developments is that T_n is essentially the value defined by the analytic continuation of some Dirichlet series at $-2n-1$, and the study of the identities (1.2) is thus closely connected to algebraic and analytic number theory, in addition to their hypergeometric q-series nature and resurgence aspect [11]. Some similar pairs of relations such as (1.2) are now known; see, for example, [5, 26, 39].

Here and throughout this paper, the asymptotic relation

$$a_n = b_n\big(1+O\big(n^{-1}\big)\big) \text{ is abbreviated as } a_n \simeq b_n. \tag{1.3}$$

Subsequently in [8], Bringmann–Li–Rhoades established the asymptotic approximation to the number of primitive row-Fishburn matrices with entries summing to n,

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big((1+z)^j-1\big) \simeq c\rho^n\, n^{n+\frac{1}{2}}, \quad \text{with} \quad (c,\rho) := \Big(\tfrac{12}{\pi^{3/2}}\,e^{-\frac{\pi^2}{24}}, \tfrac{12}{e\pi^2}\Big), \tag{1.4}$$

which confirmed a conjecture by Jelínek (Conjecture 5.3 of [31]). Here a primitive row-Fishburn matrix is a binary upper-triangular matrix without zero rows. Their proof of deriving (1.4) relies on various properties of the function ($\sigma(q)$ in [8])

$$R(q) := \sum_{k\geqslant 0} \frac{q^{\frac{1}{2}k(k+1)}}{(1+q)\cdots(1+q^k)} = 1 + \sum_{k\geqslant 0}(-1)^k q^{k+1} \prod_{1\leqslant j\leqslant k}(1-q^j), \tag{1.5}$$

first appeared in Ramanujan's lost notebook, with many unusual properties discovered since Andrews's paper [2]; see [3, 10] and A003406 for more information. Very roughly, since

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big((1+z)^j-1\big) = \frac{(-1)^n}{2}[z^n]R(1-z)$$

(according to Equation (2.3) of [8]) and $e^{-z} = 1 - z + O(|z|^2)$ for small $|z|$, the approach begins by working out the asymptotics of $[z^n]R\big(e^{-z}\big)$. The bridge between $[z^n]R(1-z)$ and $R\big(e^{-z}\big)$ can then be linked through a direct change of variables and straightforward arguments because z is very close to zero (the arguments used in [45] and [8] relying instead on the asymptotics of the Stirling numbers of the first kind); see Section 4.2 for more details.

The asymptotics of $[z^n]R\big(e^{-z}\big)$ is derived by first defining the Dirichlet series

$$D(s) := \sum_{n\geqslant 1} n^{-s}[q^{n-1}]R\big(q^{24}\big),$$

which can be meromorphically continued into the whole s-plane. Since, by standard Mellin transform techniques (see, e.g., [19]),

$$[z^n]e^{-z}R\big(e^{-24z}\big) = \frac{(-1)^n}{n!}\,D(-n),$$

the crucial asymptotics of $D(-n)$ needed is then based on the relation

$$D(-s) = \frac{12\sqrt{2}}{\pi^2}\Big(\frac{288}{\pi^2}\Big)^s\Gamma(1+s)^2\bigg(\Big(\sin\frac{\pi s}{2}\Big)^2 L_+(1+s) + \Big(\cos\frac{\pi s}{2}\Big)^2 L_-(1+s)\bigg); \tag{1.6}$$

here $L_\pm$ are L-series defined from R and a closely related q-series; see [8, 10] and Section 4 for more details.

However, a lot more on the asymptotics and statistics of Fishburn matrices and related structures has remained unknown, which includes several conjectures and open problems [8, 31, 41, 45] that are of great interest to the combinatorics and modular-form community. As apparently general asymptotic techniques are still lacking, we aim to address this gap by developing a *direct, self-contained* approach to derive (1.1) and (1.4) in a systematic way without relying on any functional equations (satisfied by Dirichlet series) or identities such as (1.2) that are not available in more general contexts with a similar sum-of-finite-product form for the generating functions.

Our approach is based instead on a fine, double saddle-point analysis and has the additional advantages of being applicable to a large number of problems whose (formal) generating functions assume a similar form, some of which are listed as follows.

– Derive the asymptotics of Fishburn and row-Fishburn matrices whose entries belong to any multiset of nonnegative integers containing 0; in particular, (1.1) and (1.4) by

Zagier [45] and Bringmann–Li–Rhoades [7], respectively, are reproved; our scheme is also applicable to more than two dozens of other OEIS sequences; see Section 6.

– Prove a conjecture of Jelínek in [31] concerning the asymptotics of self-dual Fishburn matrices; see Corollary 27.
– Establish the limit distributions of some typical statistics in random Fishburn matrices, which solves particularly an open problem raised by Bringmann–Li–Rhoades [8] and Jelínek [31]; see Theorem 22 and Section 7.
– Determine the typical shapes of random Fishburn matrices, which exhibit an unexpected change of limit laws from normal to Poisson when the smallest nonzero entry is 2; see Theorem 32 and Section 8.

Our approach is best illustrated through the prototypical (rational) sequence

$$a_n := [z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(e^{jz}-1\right) = \frac{(-1)^n}{2}[z^n]R\left(e^{-z}\right), \tag{1.7}$$

where R is defined in (1.5), for which we will show *inter alia* that

$$a_n = c\rho^n n^{n+\frac{1}{2}}\left(1+\frac{\nu_1}{n}+\frac{\nu_2}{n^2}+O\left(n^{-3}\right)\right), \quad \text{with} \quad (c,\rho) := \left(\tfrac{12}{\pi^{3/2}}, \tfrac{12}{e\pi^2}\right), \tag{1.8}$$

where $\nu_1 = \frac{24-\pi^2}{288}$ and $\nu_2 = \frac{1}{2}\nu_1^2$; see (4.1) for an asymptotic expansion. Here the integer sequence $\{a_n n!\}$ corresponds to A158690 in the OEIS. *Throughout this paper, we do not distinguish between ordinary and exponential generating functions, and focus only on the large-n asymptotics of the coefficients, so whether the sequence is integer or not is immaterial for our purposes.*

Once the asymptotics (1.8) is available, we extend our approach to sequences of the form

$$[z^n]\sum_{k\geqslant 0} d(z)^{k+\omega_0}\prod_{1\leqslant j\leqslant k}\left(e(z)^{j+\omega}-1\right)^{\alpha}, \tag{1.9}$$

for $\alpha\in\mathbb{Z}^+$ and $\omega_0,\omega\in\mathbb{C}$. Here the generating functions $d(z)$ and $e(z)$ are analytic at $z=0$ and satisfy $d(0)>0, e(0)=1$ and $e'(0)\neq 0$.

Our result (Theorem 18) for the general form (1.9) will not only be applied to derive the large-n asymptotics of many sequences in the literature and the OEIS (see Section 6), but also be sufficient to re-derive (1.1) because of the identity (in the sense of formal power series) due to Andrews and Jelínek [4]

$$\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(1-(1-z)^j\right) = \sum_{k\geqslant 0}(1-z)^{-k-1}\prod_{1\leqslant j\leqslant k}\left((1-z)^{-j}-1\right)^2.$$

This and other examples of similar types are collected in Section 6.

In addition to its usefulness in univariate asymptotics, our formulation (1.9) is also effective in dealing with the limiting distributions of various statistics (bivariate asymptotics) on random Fishburn matrices with or without restriction on their entries, which describe the typical shape of random Fishburn matrices.

More precisely, we consider upper-triangular matrices whose entries belong to Λ, a multiset of nonnegative integers with the generating function $\Lambda(z) = 1+\lambda_1 z+\lambda_2 z^2+\cdots$. We then define two classes of matrices: (i) Λ-*row-Fishburn* ones without zero row, and (ii) Λ-*Fishburn* ones without zero row or zero column. The statistics examined and their limit laws are summarized in Table 1, where we assume a uniform distribution on the set of all possible such matrices with the same entry-sum.

$\Lambda(z)$ analytic $\lambda_1 > 0$	Random Λ-row-Fishburn matrices	Random Λ-Fishburn matrices
First row sum	Zero-Truncated-Poisson$(\log 2)$	Normal$(\log n, \log n)$
Diagonal sum	Normal$(\log n, \log n)$	Normal$(2\log n, 2\log n)$
$\frac{1}{2}\big(n - \#(1\text{s})\big)$	$\begin{cases} \text{Poisson}\big(\frac{\lambda_2\pi^2}{12\lambda_1^2}\big), & \text{if } \lambda_2 > 0 \\ \text{degenerate}, & \text{if } \lambda_2 = 0 \end{cases}$	$\begin{cases} \text{Poisson}\big(\frac{\lambda_2\pi^2}{6\lambda_1^2}\big), & \text{if } \lambda_2 > 0 \\ \text{degenerate}, & \text{if } \lambda_2 = 0 \end{cases}$

Table 1. The various asymptotic distributions of the three statistics in large random Λ-row-Fishburn and Λ-Fishburn matrices with entries belonging to a given multiset of nonnegative integers Λ (containing 0 exactly once and 1 at least once). Here n is the sum of all entries in the matrix.

In particular, when Λ is the set of nonnegative integers, the first row-size in random Fishburn matrices also arises in many different contexts under different guises, the first being in the form of leftmost chord in regular linearized chord diagrams [41]; see Section 2.3 for more information. Our limit results thus have many different interpretations and implications.

The proof of these limit laws requires the full power of our setting (1.9) where some parameters or coefficients are themselves complex variables, as well as the Quasi-Powers Framework (see [20, 27, 28]), which is a simple synthetic scheme for deriving asymptotic normality and some of its quantitative refinements.

From Table 1 we see that in a typical random Λ-Fishburn matrix (when all matrices of the same entry-sum are equally likely), entries equal to 1 are ubiquitous, those to 2 appear like a Poisson distribution, and the rest is asymptotically negligible. Thus such random matrices have little variation as far as the distribution of entries is concerned. In other words, a random Λ-Fishburn matrix is asymptotically close to its primitive counterpart in which only 0 and 1 are allowed as entries. Regarding a Fishburn matrix of size n as an integer partition of n arranged on upper-triangular square matrices without zero row or column, we see that the number of 1s in random Fishburn matrices is very different from the number of 1s in random integer partitions, which has an exponential distribution in the limit.

What happens if we drop the omnipresent entry 1, assuming that all Λ-Fishburn matrices (of the same size) whose smallest nonzero entries are 2 are equally likely? The resulting random matrices turn out to be more interesting, exhibiting less expected behaviors. More precisely, we extend further our study in Section 8 to the situation when $\lambda_1 = 0$ and $\lambda_2 > 0$ in Λ-Fishburn matrices, which has a very similar analytic context to the self-dual (or persymmetric) Fishburn matrices when $\lambda_1 > 0$; the latter was considered in [31] for the cases when $\Lambda = \{0, 1\}$ and $\Lambda = \mathbb{Z}_{\geqslant 0}$. We adopt the same framework (1.9) and address the asymptotics when $\lambda_1 = 0$ and $\lambda_2 > 0$. Such a formulation is, as in the case of $\lambda_1 > 0$, not only useful for the asymptotic enumeration of matrices of large size, but also practical in characterizing the finer stochastic behaviors of the random matrices, whether they are Fishburn with $\lambda_1 = 0$ and $\lambda_2 > 0$ or self-dual Fishburn with $\lambda_1 > 0$.

$\Lambda(z)$ analytic	Random Λ-Fishburn matrices with $\lambda_{2i-1}=0$ for $1 \leqslant i \leqslant m$ and $\lambda_2, \lambda_{2m+1} > 0$	Random self-dual Λ-Fishburn matrices with 1s ($\lambda_1 > 0$)
First row sum	$\text{Normal}(\log n, \log n)$	$\text{Normal}(\log n, \log n)$
Diagonal sum	$\text{Normal}(2\log n, 2\log n)$	$2 \cdot \text{Normal}(\log n, \log n)$
# smallest nonzero entries	$\begin{cases} \frac{1}{2}n - \frac{3}{2} \cdot \text{Normal}(\tau\sqrt{n}, \tau\sqrt{n}), \\ \quad \text{if } m = 1 \\ \frac{1}{2}n - 2 \cdot \text{Poisson}(\frac{\lambda_4 \pi^2}{6\lambda_2^2}), \\ \quad \text{if } m \geqslant 2, \lambda_4 > 0, n \text{ even} \\ \frac{1}{2}(n-2m-1) - 2 \cdot \text{Poisson}(\frac{\lambda_4 \pi^2}{6\lambda_2^2}), \\ \quad \text{if } m \geqslant 2, \lambda_4 > 0, n \text{ odd} \\ \text{degenerate, if } \lambda_3 = \lambda_4 = 0 \end{cases}$	$\begin{cases} n - 2 \cdot \text{Poisson}\big(\frac{\lambda_2}{\lambda_1}\log 2\big) \\ \quad * \, 4 \cdot \text{Poisson}\big(\frac{\lambda_2 \pi^2}{12\lambda_1^2}\big), \\ \quad \text{if } \lambda_2 > 0 \\ \text{degenerate, if } \lambda_2 = 0 \end{cases}$

Table 2. The asymptotic distributions of the three statistics in large random Λ-Fishburn and self-dual Λ-Fishburn matrices with entries belonging to a given multiset of nonnegative integers Λ (containing 0 exactly once and with or without 1s). Here n is the sum of all entries in the matrix, and $\tau := \frac{\lambda_3 \pi}{2\sqrt{3}\lambda_2^{3/2}}$. The symbol $X * Y$ stands for the convolution of two distributions.

While the logarithmic behaviors in the first row sum and the diagonal sum are similar as in Table 1, the limit laws of the occurrences of the smallest nonzero entries now behave differently, notably in the case when 2 is the smallest nonzero entry. Roughly, the periodicity resulted from the prevalent entries 2 in the class of Λ-Fishburn matrices without entries 1 does change drastically the behavior of non-smallest positive entries, namely, the limit law for the sum of non-smallest positive entries changes from a bounded Poisson distribution to a normal distribution with mean and variance both asymptotic to $\tau\sqrt{n}$ (or indeed a Poisson distribution with unbounded mean $\tau\sqrt{n}$; see Section 8.2).

Our formulation and results include as a special case the asymptotic approximation to self-dual Fishburn numbers (8.5), confirming another conjecture in [31]; see Sections 5 and 8.

This paper is structured as follows. In the next section, we outline the background on Fishburn matrices, and then derive the generating functions that will be analyzed in later sections. Then we describe the saddle-point method in detail in Section 3 which will then be used and modified throughout this paper, with the finer asymptotic expansions briefly discussed in Section 4. The general framework (1.9) is examined in detail in Section 5 by extending the saddle-point analysis of Section 3. Asymptotics of restricted Fishburn matrices as well as other univariate examples are collected and discussed in Section 6. We then turn to bivariate asymptotics in Section 7 and study the asymptotic distributions of statistics on random Fishburn matrices such as those given in Table 1. The extension of (1.9) to the case when $[z]e(z) = 0$ and $[z^2]e(z) > 0$ is examined in Section 8, together with univariate and bivariate applications (as shown in Table 2). We then conclude this paper in Section 9 with some perspectives on how our approach may be further extended to other frameworks.

Notations. As mentioned at the beginning of this section, (c, ρ) is used generically and will always be locally defined. Other generic and mostly local symbols include c_i, $c(\cdot)$, ε, f, and a_n; their values will be specified whenever ambiguities may occur. Furthermore, the notation $a_n \asymp b_n$ means that the ratio a_n/b_n of the two sequences remains bounded and nonzero as n tends to infinity.

2. Fishburn matrices and related combinatorial objects

We describe Fishburn matrices in this section, together with some of their variants and generalizations. We also derive the bivariate generating functions for some statistics that will be examined in more detail in later sections.

In what follows, the *size* of a matrix is defined to be the sum of all its entries. Similarly, the size of a row or a column or the diagonal is the respective sum.

Definition 1 (Fishburn matrix)**.** A Fishburn matrix is an upper-triangular square matrix with nonnegative integer entries such that no row and no column consists solely of zeros.

For example, all 15 Fishburn matrices of size 4 are depicted in Figure 2.1.

$$(4)\ \begin{pmatrix}1&2\\&1\end{pmatrix}\begin{pmatrix}2&1\\&1\end{pmatrix}\begin{pmatrix}1&1\\&2\end{pmatrix}\begin{pmatrix}2&0\\&2\end{pmatrix}\begin{pmatrix}3&0\\&1\end{pmatrix}\begin{pmatrix}1&0\\&3\end{pmatrix}$$

$$\begin{pmatrix}1&1&0\\&1&0\\&&1\end{pmatrix}\begin{pmatrix}1&0&1\\&1&0\\&&1\end{pmatrix}\begin{pmatrix}1&0&0\\&1&1\\&&1\end{pmatrix}\begin{pmatrix}2&0&0\\&1&0\\&&1\end{pmatrix}\begin{pmatrix}1&0&0\\&2&0\\&&1\end{pmatrix}\begin{pmatrix}1&0&0\\&1&0\\&&2\end{pmatrix}\begin{pmatrix}1&1&0\\&0&1\\&&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\&1&0&0\\&&1&0\\&&&1\end{pmatrix}$$

Figure 2.1. All 15 Fishburn matrices of size 4. The occurrence of 1 is seen to be predominant.

As a succinct representation tool for interval orders (see [18, 23]), Fishburn matrices (named so in [9]; called IO-matrices in [23], characteristic matrices in [17, 18], and composition matrices in [12]) offer not only algorithmic but also combinatorial advantages, and over the years their study was largely enriched by the corresponding developments in combinatorial enumeration and bijections, following notably the paper by Bousquet-Mélou–Claesson–Dukes–Kitaev [6]. In particular, the useful database OEIS [37] played a key role in linking various structures in different areas, some of which will be briefly described later.

Closer to our interest here, the enumeration of Fishburn matrices of a given *dimension* was already investigated in the early papers [1, 23], and recursive algorithms were later proposed for computing matrices of a given size (see e.g. [25, 41]), culminating in the definitive work of Zagier [45], where, through the use of generating functions, effective asymptotic approximations (1.1) for Fishburn matrices of large size are derived.

2.1. **Fishburn matrices and their variants.** Recall that the Fishburn numbers (A022493) count Fishburn matrices of a given size and can be computed by the generating function

$$\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(1-(1-z)^j\right) = 1 + z + 2z^2 + 5z^3 + 15z^4 + 53z^5 + 217z^6 + \cdots .$$

From a combinatorial viewpoint, the Fishburn numbers also enumerate several seemingly unrelated structures, some of which are listed as follows; see [6, 9, 13, 14, 16, 18, 21, 31, 34, 36, 44] for the bijective and algebraic proofs of these equinumerosity.

- *Ascent sequences of length* n, which are sequences of nonnegative integers $(x_1, \ldots, x_n)$ such that for each i, $0 \leqslant x_i \leqslant 1 + |\{j : 1 \leqslant j \leqslant i-2 \text{ and } x_j < x_{j+1}\}|$.
- $(\mathbf{2}-\mathbf{1})$*-avoiding inversion sequences of length* n: these are sequences $(x_1, \ldots, x_n)$ such that $0 \leqslant x_i < i$ and there exists *no* $i < j$ such that $x_i = x_j + 1$.
- $(2|3\bar{1})$*-avoiding permutations of* n *elements*, which are permutations π without subsequence $\pi_i\pi_{i+1}\pi_j$ such that $\pi_i - 1 = \pi_j$ and $\pi_i < \pi_{i+1}$.
- $(\mathbf{2}+\mathbf{2})$*-free posets of* n *elements*: these are posets $(P, \prec)$ with *interval representations*, namely, for each $x \in P$, a real closed interval $[\ell_x, r_x]$ is associated to x such that $x \prec y$ in P exactly when $r_x < \ell_y$.
- *Stoimenow matchings of length* $2n$: A matching of the set $[2n] = \{1, 2, \ldots, 2n\}$ is a partition of $[2n]$ into subsets (called *arcs*) of size exactly two. A Stoimenow matching is a matching without nested pair of arcs such that either the openers or the closers are next to each other.
- *Regular linearized chord diagrams of length* $2n$: A regular linearized chord diagram is a fixed-point free involution τ on the set $[2n]$ such that $[i, i+1] \subset [\tau(i+1), \tau(i)]$ whenever $\tau(i+1) < \tau(i)$.

Two variants of Fishburn matrices, *row-Fishburn matrices* and *self-dual Fishburn matrices* were studied by Jelínek [31] during his study on refined enumeration of self-dual interval orders. *Row-Fishburn matrices* are upper-triangular ones with nonnegative integer entries such that no row is composed solely of zeros. The corresponding generating function satisfies

$$\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left((1-z)^{-j}-1\right) = 1 + z + 3z^2 + 12z^3 + 61z^4 + 380z^5 + 2815z^6 + \cdots, \tag{2.1}$$

where the coefficient of z^n equals the number of row-Fishburn matrices of size n.

Furthermore, a matrix is *primitive* if all entries are either 0 or 1. Substituting z by $\frac{z}{1+z}$ leads to the generating function for the *primitive row-Fishburn number* (A179525)

$$\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left((1+z)^{j}-1\right) = 1 + z + 2\,z^2 + 7\,z^3 + 33\,z^4 + 197\,z^5 + 1419\,z^6 + \cdots. \tag{2.2}$$

Reversely, (2.1) can be obtained from (2.2) by substituting z with $\frac{z}{1-z}$.

On the other hand, a Fishburn matrix is *self-dual* if it is persymmetric, or symmetric with respect to the northeast-southwest diagonal. The generating function of primitive self-dual Fishburn matrices counted by the size (see also [31]) is

$$\sum_{k\geqslant 0}(1+z)^{k+1}\prod_{1\leqslant j\leqslant k}((1+z^2)^j-1) = 1 + z + z^2 + 2z^3 + 3z^4 + 6z^5 + 13z^6 + \cdots,$$

and the one of all self-dual Fishburn matrices is

$$\sum_{k\geqslant 0}(1-z)^{-k-1}\prod_{1\leqslant j\leqslant k}((1-z^2)^{-j}-1) = 1 + z + 2z^2 + 3z^3 + 7z^4 + 13z^5 + 33z^6 + \cdots,$$

so that 7 out of the 15 Fishburn matrices of size 4 are self-dual, as can be readily checked with Figure 2.1.

2.2. **Fishburn matrices with entry restrictions.** We now extend the matrices by allowing more flexible entries. Let Λ be a multiset of nonnegative integers with the generating function

$$\Lambda(z) := 1 + \sum_{\lambda\in\Lambda} z^{\lambda} = 1 + \lambda_1 z + \lambda_2 z^2 + \cdots . \tag{2.3}$$

Assume *throughout this paper that* $\Lambda(z)$ *is analytic at* $z = 0$*, and except in Sections 8.1, 8.3 and 8.4 assume that* $\lambda_1 > 0$, so that $\{0, 1\} \subseteq \Lambda$.

Definition 2 (Λ-Fishburn matrix)**.** An upper-triangular matrix is called a Λ-*Fishburn* matrix if every row and column has non-zero size, and all entries lie in the set Λ.

Definition 3 (Λ-row-Fishburn matrix)**.** An upper-triangular matrix is called a Λ-*row-Fishburn* matrix if all entries lie in the set Λ without zero row.

Proposition 1. *The number of* Λ*-row-Fishburn matrices of size* n *is given by*

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(\Lambda(z)^j - 1\big), \tag{2.4}$$

and that of Λ*-Fishburn matrices by*

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(1-\Lambda(z)^{-j}\big) = [z^n]\sum_{k\geqslant 0}\Lambda(z)^{k+1}\prod_{1\leqslant j\leqslant k}\big(\Lambda(z)^j - 1\big)^2. \tag{2.5}$$

Proof. The first generating function (2.4) follows from the definition of Λ-row-Fishburn matrices. For a Λ-row-Fishburn matrix of dimension k, and for any j, $1 \leqslant j \leqslant k$, the generating function of the $(k-j+1)$-st row counted by the size (variable z) is $\Lambda(z)^j - 1$. As a result, the generating function for Λ-row-Fishburn matrices of dimension k is given by $\prod_{1\leqslant j\leqslant k}(\Lambda(z)^j - 1)$. Summing over all k leads to (2.4).

On the other hand, the generating function for primitive Fishburn matrices is given by (including the constant 1 for the empty matrix; see [31])

$$\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(1-(1+z)^{-j}\big).$$

Substituting $1 + z$ by $\Lambda(z)$ yields the generating function for Λ-Fishburn matrices in the general case.

The right-hand side of the identity (2.5) follows from the following q-identity due to Andrews and Jelínek [4]

$$\begin{aligned}\sum_{k\geqslant 0} u^k \prod_{1\leqslant j\leqslant k}&\left(1-\frac{1}{(1-s)(1-t)^{j-1}}\right)\\ &= \sum_{k\geqslant 0}(1-s)(1-t)^k\prod_{1\leqslant j\leqslant k}\Big(\big(1-(1-s)(1-t)^{j-1}\big)\big(1-u(1-t)^j\big)\Big),\end{aligned} \tag{2.6}$$

by substituting $u = 1$ and $s = t = 1 - \Lambda(z)$ on both sides. ∎

2.3. **Statistics on Fishburn matrices.** The study of statistics on the Fishburn structures traces back to the work by Andresen and Kjeldsen [1] in the context of transitively directed graphs (see also [31]), where they studied the numbers of primitive Fishburn matrices counted by the dimension and by the size of the first row (with the notation $\xi(n,k)$ in [1]).

Stoimenow [41] found a recursive formula for the numbers of regular linearized chord diagrams with a given length of the leftmost chord. Subsequently, it was discovered [6, 21, 31, 35, 36, 44] that these numbers are equivalent to the following ones:

- the sum of entries in the first row (or the last column) of Fishburn matrices of size n;
- the number of minimal (or maximal) elements in $(2+2)$-free posets of size n;
- the maximal entries (or right-to-left minimal entries, or the number of zeros) in ascent sequences of length n;
- the maximal entries in $(\mathbf{2}-\mathbf{1})$-avoiding inversion sequences of length n;
- the length of the initial run of openers in Stoimenow matchings of length $[2n]$;
- the length of the initial decreasing run in $(2|3\bar{1})$-avoiding permutations of length n;
- the number of left-to-right minima (or left-to-right maxima; or right-to-left maxima) in $(2|3\bar{1})$-avoiding permutations of length n.

These statistics are classified as Stirling statistics; see [21]. In parallel, the classical Eulerian statistics have also been intensively studied but the corresponding limiting distributions were only recently solved in a subsequent paper [30].

2.4. **Bivariate generating functions for Fishburn matrices with entry restrictions.** We study the asymptotic distributions of the following three random variables on random Λ-Fishburn and Λ-row-Fishburn matrices, assuming a uniform distribution on the set of all size-n matrices: the size of the first row, the size of the diagonal, and the number of occurrences of 1.

The approach we use to characterize the corresponding limit laws relies heavily on the corresponding bivariate generating functions and our double saddle-point analysis. We derive the necessary generating functions in this subsection. We use the convention that $f(z,v)$ is the bivariate generating function for the quantity X if $[z^n v^m]f(z,v)$ denotes the number of Λ-row-Fishburn matrices of size n with $X=m$.

Proposition 2 (Statistics on Λ-row-Fishburn matrices)**.** *We have the following bivariate generating functions of Λ-row-Fishburn matrices with z marking the matrix size and v marking respectively:*

(i) the size of the first row

$$1+\sum_{k\geqslant 0}\big(\Lambda(vz)^{k+1}-1\big)\prod_{1\leqslant j\leqslant k}\big(\Lambda(z)^j-1\big), \tag{2.7}$$

(ii) the size of the (main) diagonal (or the last column)

$$\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(\Lambda(vz)\Lambda(z)^{j-1}-1\big), \quad \textit{and} \tag{2.8}$$

(iii) the number of 1s

$$\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big((\Lambda(z)+\lambda_1(v-1)z)^j-1\big). \tag{2.9}$$

Proof. Given a Λ-row-Fishburn matrix of dimension $k+1$, the generating function for the first row size (marked by vz) is $\Lambda(vz)^{k+1}-1$, the remaining k rows contributing $\prod_{1\leqslant j\leqslant k}\big(\Lambda(z)^j-1\big)$, as in the proof of (2.4). The proofs for the other two parameters are similar and thus omitted. □

For Λ-Fishburn matrices, the proof is less straightforward and we need a fine-tuned version of Jelínek's Theorem 2.1 in [31] in order to enumerate both the first row and the diagonal sizes.

Let $\mathcal{P}$ denote the set of primitive Λ-Fishburn matrices. Define first

$$\begin{aligned}&G_k(t,u,v,w,x,y)\\ &\quad:=\sum_{(M_{i,j})_{k\times k}\in\mathcal{P}} t^{M_{k,k}}u^{\sum_{2\leqslant j<k}M_{j,k}}v^{\sum_{2\leqslant j<k}M_{j,j}}w^{\sum_{1<i<j<k}M_{i,j}}x^{M_{1,k}}y^{\sum_{1\leqslant j<k}M_{1,j}},\end{aligned}$$

so that t marks the lower-right corner (which is always 1), u the size of the last column except the two ends, v the size of the (main) diagonal except the two ends, w the size of all interior cells, x the upper-right corner, and y the size of the first row except the upper-right corner.

Lemma 3. *The generating function* $F(s,t,u,v,w,x,y):=\sum_{k\geqslant 2}G_k(t,u,v,w,x,y)s^k$ *of primitive Λ-Fishburn matrices satisfies*

$$\begin{aligned}&F(s,t,u,v,w,x,y)\\ &\quad=t\sum_{k\geqslant 0}\frac{s^{k+2}y(1+x)(1+y)^k}{(1+u)(1+v)(1+w)^k-1}\prod_{0\leqslant j\leqslant k}\frac{(1+u)(1+v)(1+w)^j-1}{1+s\big((1+u)(1+w)^j-1\big)}.\end{aligned} \tag{2.10}$$

Proof. (Sketch) By definition, it is clear that $G_1=x$ and $G_k(t,u,v,w,x,y)=tG_k(1,u,v,w,x,y)$ for $k\geqslant 2$. A recursive construction of Fishburn matrices was discovered by Haxell–McDonald–Thomason [25] (also used in [31, Lemma 2.8]) where any Fishburn matrix of dimension $k+1$ is generated by conditioning on the entries in the last column of a Fishburn matrix of dimension k. Accordingly, we derive the recurrence relation

$$\begin{aligned}&G_{k+1}(1,u,v,w,x,y)\\ &\quad=G_k(u+v+uv,u,v+w+vw,w,x+y+xy,y)-vG_k(1,u,v,w,x,y)\\ &\quad=(u+v+uv)G_k(1,u,v+w+vw,w,x+y+xy,y)-vG_k(1,u,v,w,x,y),\end{aligned}$$

for $k\geqslant 2$. Then from this and the iterative arguments used in [31], we deduce (2.10); see [31] for details. □

Proposition 4 (Statistics on Λ-Fishburn matrices)**.** *We have the following bivariate generating functions of Λ-Fishburn matrices with z marking the matrix size and v marking respectively* *(i) the size of the first row (or the last column)*

$$\begin{aligned}(2.11)\qquad &\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(1-\Lambda(vz)^{-1}\Lambda(z)^{1-j}\big)\\ &\quad=\Lambda(vz)\sum_{k\geqslant 0}\Lambda(z)^k\prod_{1\leqslant j\leqslant k}\big(\big(\Lambda(vz)\Lambda(z)^{j-1}-1\big)\big(\Lambda(z)^j-1\big)\big),\end{aligned}$$

(ii) the size of the (main) diagonal

$$\begin{aligned}(2.12)\qquad &1+\Lambda(vz)+(\Lambda(vz)-1)^2\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(\Lambda(vz)-\Lambda(z)^{-j}\big)\\ &\quad=\Lambda(vz)\sum_{k\geqslant 0}\Lambda(z)^k\prod_{1\leqslant j\leqslant k}\big(\Lambda(vz)\Lambda(z)^{j-1}-1\big)^2,\quad \textit{and}\end{aligned}$$

(iii) the number of 1*s*

$$\begin{aligned}(2.13)\qquad &\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(1-(\Lambda(z)+\lambda_1(v-1)z)^{-j}\big)\\ &\quad=\sum_{k\geqslant 0}(\Lambda(z)+\lambda_1(v-1)z)^{k+1}\prod_{1\leqslant j\leqslant k}\big((\Lambda(z)+\lambda_1(v-1)z)^j-1\big)^2.\end{aligned}$$

Proof. (i) It is known that the generating function for the size of the first row (marked by v) in primitive Fishburn matrices is given by (see [21, 34])

$$\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}(1-(1+vz)^{-1}(1+z)^{1-j}).$$

Substituting $1+vz$ by $\Lambda(vz)$ and $1+z$ by $\Lambda(z)$ gives the left-hand side of (2.11), which equals the right-hand side of (2.6) after substituting $u=1$, $s=1-\Lambda(vz)$ and $t=1-\Lambda(z)$, i.e., the generating function on the right-hand side of (2.11). Alternatively, one can derive (2.11) by using (2.10), Andrew-Jelínek identity (2.6), the identity [4, Eq. (1)] and substitutions.

(ii) For the size of the diagonal, we have, again, by (2.10), the generating function

$$1+vz+F(1,v^2z,z,vz,z,z,z)=1+vz+(vz)^2\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(1+vz-(1+z)^{-j}\big).$$

The same substitutions $1+vz\mapsto\Lambda(vz)$ and $1+z\mapsto\Lambda(z)$ give the left-hand side of (2.12). Applying now (2.6) with $u=1+vz$, $s=1-(1+vz)(1+z)$ and $t=-z$, and then using the same substitutions, we obtain the right-hand side of (2.12).

(iii) The generating functions (2.13) for the number of 1s follow from substituting $\Lambda(z)$ by $\Lambda(z)+\lambda_1(v-1)z$ in (2.5).

□

3. Asymptotics of the prototype sequence A158690

Consider the sequence $a_n:=[z^n]A(z)$, where

$$(3.1)\qquad A(z):=\sum_{k\geqslant 0}A_k(z),\quad\text{with}\quad A_k(z):=\prod_{1\leqslant j\leqslant k}\big(e^{jz}-1\big),$$

which is used as the running and prototypical example of our analytic approach. The sequence $\{n!a_n\}_{n\geqslant 0}$ equals A158690 and can be generated (in addition to (3.1)) by many different forms (see [2, 8]), showing partly the diversity and structural richness of the sequence

$$\begin{aligned}
A(z) &= \sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \left(1-e^{-(2j-1)z}\right) \\
&= \sum_{k\geqslant 0} e^{-(k+1)z} \prod_{1\leqslant j\leqslant k} \left(1-e^{-2jz}\right) \\
&= \sum_{k\geqslant 0} e^{(2k+1)z} \prod_{1\leqslant j\leqslant 2k} \left(e^{jz}-1\right) \\
&= \frac{1}{2}\left(1+\sum_{k\geqslant 0} e^{(k+1)z} \prod_{1\leqslant j\leqslant k} \left(e^{jz}-1\right)\right).
\end{aligned}$$

Among these series forms, we work on (3.1) because it is simpler and $A_k(z)$ contains only positive Taylor coefficients.

Theorem 5. *Define $A(z)$ by (3.1). Then as n tends to infinity,*

$$a_n := [z^n]A(z) \simeq c\rho^n n^{n+\frac{1}{2}}, \quad \text{with} \quad (c,\rho) := \left(\tfrac{12}{\pi^{3/2}}, \tfrac{12}{e\pi^2}\right). \tag{3.2}$$

Our approach consists in computing the asymptotics of $a_{n,k} := [z^n]A_k(z)$ by the saddle-point method (see [20]) for each $1 \leqslant k < n$, and then summing $a_{n,k}$ over all k (in turn involving another application of the saddle-point method); indeed, due to high concentration near the maximum, only a small neighborhood of k near μn, $\mu := \frac{12}{\pi^2}\log 2 \approx 0.84$, will contribute to the dominant asymptotics (3.2). Thus we are in the context of a double saddle-point method.

More precisely, we begin with the expression

$$a_n = \sum_{1\leqslant k\leqslant n} a_{n,k} = \sum_{1\leqslant k\leqslant n} \frac{r^{-n}}{2\pi i}\int_{-\pi}^{\pi} e^{-in\theta} A_k(re^{i\theta})\,\mathrm{d}\theta,$$

and follow the procedures outlined below.

- Find the positive pair (k, r) such that r is the saddle-point of $A_k(z)$ and the value $r^{-n}A_k(r)$ reaches its maximum as n tends to infinity, so as to identify the terms $a_{n,k}$ that reach the maximum modulus for each fixed n; see Lemma 11.
- Once the range of $k \sim \mu n$ is identified, show, by a simple saddle-point bound for Taylor coefficients, that the contribution to a_n of $a_{n,k}$ from the range $|k-\mu n| \geqslant n^{\frac{5}{8}}$ is asymptotically negligible; see Proposition 12.
- In the central range $|k-\mu n| \leqslant n^{\frac{5}{8}}$, the integral $\int_{n^{-\frac{3}{8}}\leqslant|\theta|\leqslant\pi}$ is asymptotically negligible; see Proposition 14.
- Then inside the ranges $|k-\mu n| \leqslant n^{\frac{5}{8}}$ and $\int_{|\theta|\leqslant n^{-\frac{3}{8}}}$, compute the asymptotic approximation (3.2) by local expansions and term-by-term integration; see Section 3.5.
- These procedures can be refined to get longer expansions if desired.

For all these purposes, it turns out that a precise asymptotic approximation to $\log A_k(r)$ will largely simplify the analysis. Since we will also need asymptotics of the derivatives of

$\log A_k(r)$, we propose a complex-variable version so as to avoid repeated use of the Euler-Maclaurin formula.

3.1. **Euler-Maclaurin formula and asymptotic expansions.**

We apply the Euler-Maclaurin formula to approximate the various sums encountered in this paper, which for completeness is included as follows.

Lemma 6 (Euler-Maclaurin formula). *Assume that φ is m-times continuously differentiable over the interval $[a, b]$, $m \geqslant 1$. Then*

$$\sum_{j=a+1}^{b} \varphi(j) = \int_a^b \varphi(t)\,\mathrm{d}t + \frac{\varphi(b) - \varphi(a)}{2} + \sum_{\ell=1}^{\lfloor m/2 \rfloor} \frac{B_{2\ell}}{(2\ell)!}\big(\varphi^{(2\ell-1)}(b) - \varphi^{(2\ell-1)}(a)\big) + \frac{(-1)^{m+1}}{m!}\int_a^b \varphi^{(m)}(t) B_m(\{t\})\mathrm{d}t, \tag{3.3}$$

where $\{x\}$ denotes the fractional part of x, the B_ℓ's and the $B_n(t)$'s are Bernoulli numbers and polynomials, respectively.

When φ is infinitely differentiable (which is the case for all functions considered in this paper), we can push the expansion to any $m > 0$ depending on the required error, keeping the error term under control.

The following expansion is crucial in our saddle-point analysis. Let

$$L_k(z) := \log A_k(z) = \sum_{1 \leqslant j \leqslant k} \log\big(e^{jz} - 1\big).$$

Proposition 7. *For $k \to \infty$, we have*

$$L_k(z) = k\log\big(e^{kz} - 1\big) - \frac{I(kz)}{z} + \frac{1}{2}\log\frac{2\pi\big(e^{kz}-1\big)}{z} + \frac{z\big(e^{kz}+1\big)}{24\big(e^{kz}-1\big)} + \sum_{2 \leqslant j < m} \frac{B_{2j}}{(2j)!} \cdot \frac{z^{2j-1}e^{-kz}E_{2j-2}\big(e^{-kz}\big)}{\big(1-e^{-kz}\big)^{2j-1}} + O\big(k^{1-2m} + |z|^{2m-1}\big), \tag{3.4}$$

uniformly for $k|z| \leqslant 2\pi - \varepsilon$ when $|\arg z| \leqslant \pi - \varepsilon$, where

$$I(z) := \int_0^z \frac{t}{1 - e^{-t}}\,\mathrm{d}t,$$

and $E_n(x) = \sum_{0 \leqslant j < n} \left\langle {n \atop j} \right\rangle x^j$ denote the Eulerian polynomials.

Proof. For simplicity and for later use, we compute only the first few terms by working out $m = 2$, as the general form follows from the relation

$$\partial_z^m \log\big(e^{xz} - 1\big) = (-1)^{m-1}\frac{x^m e^{-xz}E_{m-1}\big(e^{-xz}\big)}{\big(1-e^{-xz}\big)^m} \qquad (m \geqslant 2);$$

see [42] for similar analysis.

Since $\log\bigl(e^{jz}-1\bigr)$ is undefined at $j=0$, we split the sum into two parts:

$$L_k(z) = \log k! - \sum_{1\leqslant j\leqslant k} \log\frac{j}{e^{jz}-1}.$$

By the Euler-Maclaurin formula (3.3), we find that

$$\sum_{1\leqslant j\leqslant k} \log\frac{j}{e^{jz}-1} = \int_0^k \log\frac{x}{e^{xz}-1}\,\mathrm{d}x + \frac{1}{2}\log\frac{kz}{e^{kz}-1} + \frac{1}{12}\Bigl(\frac{1}{k}+\frac{z}{2}-\frac{z}{1-e^{-kz}}\Bigr) + O\bigl(k^{-2}+|z|^2\bigr).$$

Then an integration by parts gives

$$\int_0^k \log\frac{x}{e^{xz}-1}\,\mathrm{d}x = k\log\frac{k}{e^{kz}-1} - k + \frac{I(kz)}{z}.$$

The first few terms of (3.4) then follow from this and Stirling's formula for $\log k!$.

For the error term, by (3.3) with $m=2$ and $B_2(x) = x^2-x+\frac{1}{6}$, we have

$$R_2 := \int_0^k \Bigl(\frac{z^2e^{xz}}{(e^{xz}-1)^2} - \frac{1}{x^2}\Bigr)\bigl(\{x\}^2-\{x\}+\tfrac{1}{6}\bigr)\,\mathrm{d}x.$$

If $k|z|\leqslant 1$, then R_2 is bounded above by

$$R_2 = O\Bigl(\int_0^{1/|z|} |z|^2\,\mathrm{d}x\Bigr) = O(|z|).$$

On the other hand, if $1\leqslant k|z|\leqslant 2\pi-\varepsilon$, then

$$R_2 = O\left(|z| + \int_{1/|z|}^k \left(\frac{|z|^2e^{\Re(xz)}}{|e^{xz}-1|^2}+\frac{1}{x^2}\right)\mathrm{d}x\right) = O\bigl(|z|+k^{-1}\bigr),$$

as required, where $\Re(z)$ denotes the real part of z. This proves (3.4). □

Note that

(3.5) $$I(z) := \int_0^z \frac{t}{1-e^{-t}}\,\mathrm{d}t = \frac{z^2}{2} + \operatorname{dilog}\bigl(e^{-z}\bigr),$$

where $\operatorname{dilog}(1-z) := \sum_{k\geqslant 1}\frac{z^k}{k^2}$ denotes the dilogarithm function. Also $n![z^n]\operatorname{dilog}\bigl(e^{-z}\bigr) = B_{n-1}$, the Bernoulli numbers.

The main reason of stating this complex-variable version for $L_k(z)$ is that termwise differentiation with respect to z is allowed by analyticity in a compact domain (or Cauchy's integral formula for derivatives), leading to an asymptotic expansion for all higher derivatives of $L_k(z)$; see, e.g., [38, 43]. In this way, we obtain, for example, the following approximations, which will be needed below.

Corollary 8. *Uniformly as $k\to\infty$ and $k|z|\leqslant 2\pi-\varepsilon$ when $|\arg(z)|\leqslant\pi-\varepsilon$,*

(3.6) $$zL_k'(z) = \sum_{1\leqslant j\leqslant k}\frac{jz}{1-e^{-jz}} = \frac{I(kz)}{z} + \frac{kz-1+e^{-kz}}{2\bigl(1-e^{-kz}\bigr)} + O\bigl(k^{-1}+|z|\bigr).$$

Corollary 9. *Let $m \geqslant 2$. Then*

$$
\begin{aligned}
z^m L_k^{(m)}(z) &= (-1)^{m-1} z^m \sum_{1\leqslant j\leqslant k} \frac{j^m e^{-jz} E_{m-1}\big(e^{-jz}\big)}{\big(1-e^{-jz}\big)^m} \\
&= z^m \partial_z^{m-1}\left(\frac{I(kz)}{z^2} + \frac{kz-1+e^{-kz}}{2z\big(1-e^{-kz}\big)}\right) + O\big(k^{-1}+|z|\big),
\end{aligned} \tag{3.7}
$$

uniformly as $k\to\infty$, $k|z| \leqslant 2\pi-\varepsilon$ when $|\arg z| \leqslant \pi-\varepsilon$.

In particular, we see that each $r^m L_k^{(m)}(r)$ is asymptotically of linear order when $kr = O(1)$.

3.2. **Saddle-point method. I: Identifying the central range.** A very simple uniform estimate for $a_{n,k}$ is readily obtained by the saddle-point bound for positive Taylor coefficients (see [20, Sec. VIII.2]).

Lemma 10. *For $1 \leqslant k < n$,*

$$
a_{n,k} \leqslant r^{-n} A_k(r), \tag{3.8}
$$

where $r > 0$ is chosen to be the saddle-point, namely, the unique positive solution of the equation

$$
rL_k'(r) = \frac{rA_k'(r)}{A_k(r)} = \sum_{1\leqslant j\leqslant k} \frac{jr}{1-e^{-jr}} = n. \tag{3.9}
$$

Such an r obviously exists for $n \geqslant 1$ and $1 \leqslant k < n$ because $x/(1-e^{-x}) \geqslant 1$ is monotonically increasing with $x \geqslant 0$. Also $r\to\infty$ when $k = o(n)$ and $r \to 0$ when $k \to n$. In particular, $r = 0$ when $k = n$.

The simple bound (3.8) is sufficient to give not only the factorial term n^n but also the right exponential one $\big(\frac{12}{e\pi^2}\big)^n$ in (3.2), as the following lemma shows.

Lemma 11. *For $1 \leqslant k = qn < n$,*

$$
a_{n,k} = O\big(n^{n+\frac12} e^{\phi(q,\varrho)n}\big), \quad \textit{with} \quad \phi(q,\varrho) := -\log\varrho + q\log(e^{q\varrho}-1) - 1, \tag{3.10}
$$

subject to the condition $I(q\varrho) = \varrho$. The maximum value of $\phi(q,\varrho)$ for $q \in [0,1]$ is reached when

$$
(q,\varrho) = (\mu,\xi) := \left(\frac{12}{\pi^2}\log 2, \frac{\pi^2}{12}\right), \quad \textit{with} \quad e^{\phi(\mu,\xi)} = \frac{12}{e\pi^2}. \tag{3.11}
$$

Proof. We begin with the first-order approximation to $rL_k'(r)$ already derived in (3.6) with the saddle-point $|z| = r = \frac{\varrho}{n}$ and $k = qn$,

$$
\frac{\varrho}{n} L_k'\Big(\frac{\varrho}{n}\Big) \sim \frac{n}{\varrho}\, I(q\varrho),
$$

which leads to the approximate saddle-point equation $I(q\varrho) = \varrho$. Furthermore, we have, by (3.4),

$$
\log\big(r^{-n}A_k(r)\big) = n\log n + n\phi(q,\varrho) + \tfrac12 \log n + O(1),
$$

where $\phi(q,\varrho) := -\log\varrho + q\log(e^{q\varrho}-1) - 1$, in view of $I(q\varrho) = \varrho$. We next look for the pair of (q,ϱ) such that the maximum value of $\phi(q,\varrho)$ is reached.

The only positive solution pair of the equations $\partial_q \phi(q,\varrho) = 0$ and $I(q\varrho) = \varrho$, or, equivalently,

$$\log\bigl(e^{q\varrho} - 1\bigr) = 0 \quad \text{and} \quad I(q\varrho) = \varrho,$$

is given by (3.11), which is one of the sources of the ubiquitous factor $\frac{\pi^2}{6} = \zeta(2)$ in this paper. It remains to prove $\phi(q,\varrho)$ is maximal for $q \in [0,1]$ only when (3.11) occurs. Now, by viewing $w = w(q)$ as a function of q, we see that, when (q,w) satisfies the condition $I(qw) = w$,

$$\partial_q^2 \phi(q,w) = \frac{w}{1 - q^2 w - e^{-qw}}.$$

We now prove that $\partial_q^2 \phi(q,w) < 0$ for all pairs (q,w) such that $I(qw) = w$. First, the function $t \mapsto \frac{t}{1-e^{-t}}$ is motononically increasing for $t \geqslant 0$; then, with $w \neq 0$,

$$w = \int_0^{qw} \frac{t}{1-e^{-t}}\,\mathrm{d}t < \frac{qw}{1-e^{-qw}} \cdot qw = \frac{q^2w^2}{1-e^{-qw}},$$

implying that

$$w\Bigl(1 - \frac{q^2 w}{1-e^{-qw}}\Bigr) = \frac{w(1 - q^2 w - e^{-qw})}{1-e^{-qw}} < 0.$$

Thus the function $q \mapsto \partial_q^2 \phi(q,w)$ is always negative for all pairs (q,w) such that $I(qw) = w$, showing that $q \mapsto \phi(q,w)$ is concave downward when w satisfies $I(qw) = w$; see Figure 3.1. This proves the lemma. $\square$

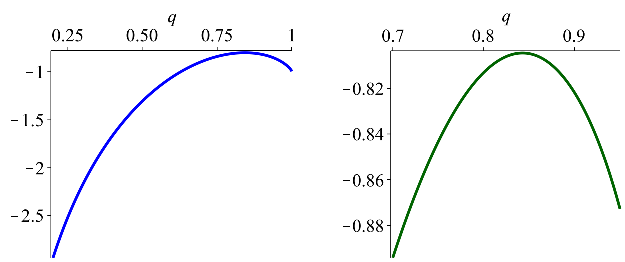


Figure 3.1. The concavity of the function $\phi(q,\varrho)$ when $\varrho = \varrho(q)$ satisfies $I(q\varrho) = \varrho$ for $q \in [0.2, 1]$ (left) and $q \in [0.7, 0.95]$ (right).

3.3. **Saddle-point method. II: Negligibility of summands outside the central range.** Define

$$\sigma := \pi^{-2}\sqrt{6\bigl(24(\log 2)^2 - \pi^2\bigr)} \approx 0.31988. \tag{3.12}$$

Proposition 12. *Write $k = \mu n + x\sigma\sqrt{n}$ where μ and σ are given in (3.11) and (3.12), respectively. Then uniformly for $x = o\bigl(n^{\frac{1}{6}}\bigr)$,*

$$a_{n,k} = O\bigl(\rho^n n^{n+\frac{1}{2}} e^{-\frac{1}{2}x^2}\bigr), \quad \textit{with} \quad \rho := \tfrac{12}{e\pi^2}. \tag{3.13}$$

In particular when $k_\pm := \mu n \pm \sqrt{2}\,\sigma n^{\frac{5}{8}}$,

$$\Biggl(\sum_{1 \leqslant k < k_-} + \sum_{k_+ < k \leqslant n}\Biggr) a_{n,k} = O\bigl(\rho^n n^{n+\frac{3}{2}} e^{-n^{\frac{1}{4}}}\bigr). \tag{3.14}$$

Proof. Assume first

$$q := \mu + \frac{\sigma x}{\sqrt{n}}, \tag{3.15}$$

where μ is defined in Lemma 11 but the value of σ given in (3.12) is (yet) unknown (and will be specified by the following procedure). Substituting this q into the saddle-point equation (3.9), as approximated by (3.6), and solving asymptotically for ϱ, we then obtain

$$\varrho = \xi + \frac{\xi_1 x}{\sqrt{n}} + \frac{\xi_2 + \xi_3 x^2}{n} + O\Big(\frac{|x| + |x|^3}{n^{\frac{3}{2}}}\Big), \tag{3.16}$$

where, with $\tau := 2(\log 2)^2 - \frac{\pi^2}{12}$

$$\xi_1 := -\tfrac{\pi^4 \log 2}{72\,\tau}, \quad \xi_2 := -\tfrac{\pi^4(2\log 2 - 1)}{288\,\tau}, \quad \text{and} \quad \xi_3 := \tfrac{\pi^6(288\tau^2 + (\log 2)\pi^4 + 24\pi^2\tau - \pi^4)}{248832\,\tau^3}. \tag{3.17}$$

Then we substitute the expansions (3.15) and (3.16) into $\phi(q, \varrho)$ (defined in Lemma 11), giving

$$\phi(q,\varrho) = -\log\frac{\pi^2}{12} - 1 + \frac{1}{n}\Big(\frac{1}{2} - \log 2 - \frac{\pi^4\sigma^2x^2}{144\tau}\Big) + O\Big(\frac{|x| + |x|^3}{n^{\frac{3}{2}}}\Big).$$

So if we take $\sigma^2 := 72\pi^{-4}\tau$ (which is identical to the expression (3.12)), then we see that

$$e^{n\phi(q,\varrho)} = \frac{\sqrt{e}}{2}\left(\frac{12}{e\pi^2}\right)^n e^{-\frac{1}{2}x^2}\left(1 + O\Big(\frac{|x| + |x|^3}{\sqrt{n}}\Big)\right),$$

uniformly for $x = o\big(n^{\frac{1}{6}}\big)$. This, together with (3.10) and Lemma 11, proves (3.13).

By monotonicity of $\phi(q, w)$ (see Lemma 11), the left-hand side of (3.14) is bounded from above by $(na_{n,k_-} + na_{n,k_+})$. In consequence, (3.14) follows from (3.13) with $x = \sqrt{2}\,n^{\frac{1}{8}}$. ☐

3.4. **Saddle-point method. III: Negligibility of integrals away from zero.** We now show that in the remaining sum ($k_\pm$ defined in Proposition 12)

$$\sum_{k_- \leqslant k \leqslant k_+} \frac{r^{-n}}{2\pi}\int_{-\pi}^{\pi} e^{-in\theta} A_k(re^{i\theta})\,\mathrm{d}\theta,$$

the integral over the range $\theta_0 \leqslant |\theta| \leqslant \pi$, $\theta_0 := 6n^{-\frac{3}{8}}$ is asymptotically negligible. Such a θ_0 is always chosen so that $n\theta_0^2 \to \infty$ and $n\theta_0^3 \to 0$; see [20]. We begin with a uniform bound for $|A_k(z)|$.

Lemma 13. *Let $\theta := \arg(z)$. Then, uniformly for $|z| > 0$ and $|\theta| \leqslant \pi$,*

$$|A_k(z)| \leqslant A_k(|z|)\exp\left(-\frac{k(k+1)|z|\,\theta^2}{2\pi^2}\right), \quad (k = 1, 2, \dots). \tag{3.18}$$

Proof. The uniform bound (3.18) is a direct consequence of the inequality (see [40, Appendix])

$$\big|e^z - 1\big| \leqslant \big(e^{|z|} - 1\big)e^{-|z|\theta^2/\pi^2}, \quad (|\theta| \leqslant \pi). \tag{3.19}$$

This is proved as follows. First

$$\big|e^z - 1\big| = \big|e^{\frac{1}{2}z}\big|\big|e^{\frac{1}{2}z} - e^{-\frac{1}{2}z}\big| \leqslant e^{\frac{1}{2}|z|\cos\theta}\big(e^{\frac{1}{2}|z|} - e^{-\frac{1}{2}|z|}\big) = \big(e^{|z|} - 1\big)e^{-\frac{1}{2}|z|(1-\cos\theta)},$$

where the inequality results from the fact that $[t^n]\big(e^t - e^{-t}\big) \geqslant 0$ for all $n \geqslant 0$. Then (3.19) follows from the elementary inequality $1 - \cos\theta \geqslant \frac{2}{\pi^2}\theta^2$ for $|\theta| \leqslant \pi$. $\square$

Proposition 14. *Define $k_\pm := \mu n \pm \sqrt{2}\,\sigma n^{\frac{5}{8}}$ as in Proposition 12 and $\theta_0 := 6n^{-\frac{3}{8}}$. Then,*

$$\sum_{k_- \leqslant k \leqslant k_+} \frac{r^{-n}}{2\pi} \int_{\theta_0 \leqslant |\theta| \leqslant \pi} e^{-in\theta} A_k(re^{i\theta})\,\mathrm{d}\theta = O\big(\rho^n n^{n-\frac{1}{8}} e^{-n^{\frac{1}{4}}}\big), \quad \textit{with} \quad \rho := \frac{12}{e\pi^2}. \tag{3.20}$$

Proof. By (3.18) with $z := re^{i\theta}$,

$$\sum_{k_- \leqslant k \leqslant k_+} \frac{r^{-n}}{2\pi} \int_{\theta_0 \leqslant |\theta| \leqslant \pi} e^{-in\theta} A_k(re^{i\theta})\,\mathrm{d}\theta = O\bigg(\sum_{k_- \leqslant k \leqslant k_+} r^{-n} A_k(r) \int_{\theta_0}^{\infty} e^{-\frac{k^2 r\theta^2}{2\pi^2}}\,\mathrm{d}\theta \bigg).$$

Now, with $k \sim \mu n$ ($k_- \leqslant k \leqslant k_+$) and $rn \sim \xi$ (see (3.11)), we then have

$$\int_{\theta_0}^{\infty} e^{-\frac{k^2 r\theta^2}{2\pi^2}}\,\mathrm{d}\theta = O\Big(\frac{n^{\frac{3}{8}}}{k^2 r}\, e^{-\frac{k^2 r}{2\pi^2 n^{3/4}}} \Big) = O\big(n^{-\frac{5}{8}} e^{-\frac{216(\log 2)^2}{\pi^4}(1+o(1)) n^{\frac{1}{4}}} \big).$$

Note that $\frac{216(\log 2)^2}{\pi^4} \approx 1.065 > 1$. Then (3.20) follows from (3.13). $\square$

3.5. **Saddle-point method. IV: Proof of Theorem 5.** From the two estimates (3.14) and (3.20), we have, with $\rho = \frac{12}{e\pi^2}$ and $\theta_0 := 6n^{-\frac{3}{8}}$,

$$a_n = \sum_{k_- \leqslant k \leqslant k_+} r^{-n} A_k(r) \cdot \frac{1}{2\pi} \int_{-\theta_0}^{\theta_0} e^{-in\theta} \frac{A_k(re^{i\theta})}{A_k(r)}\,\mathrm{d}\theta + O\big(\rho^n n^{n+\frac{3}{2}} e^{-n^{\frac{1}{4}}}\big). \tag{3.21}$$

We begin by evaluating asymptotically the integral .

Lemma 15. *If $k = \mu n + x\sigma\sqrt{n}$, where μ and σ are given in (3.11) and (3.12), respectively, then*

$$J_I := \frac{1}{2\pi} \int_{-\theta_0}^{\theta_0} e^{-in\theta} \frac{A_k(re^{i\theta})}{A_k(r)}\,\mathrm{d}\theta \simeq \frac{\sqrt{3}}{\pi^{3/2}\sigma}\, n^{-\frac{1}{2}}, \tag{3.22}$$

uniformly for $x = o\big(n^{\frac{1}{6}}\big)$.

Proof. Expand $L_k(re^{i\theta})$ in θ:

$$\log \frac{A_k\big(re^{i\theta}\big)}{A_k(r)} = L_k\big(re^{i\theta}\big) - L_k(r) := \sum_{j \geqslant 1} \frac{\upsilon_j(r)}{j!} (i\theta)^j.$$

First of all, $\upsilon_1(r) = \frac{rA_k'(r)}{A_k(r)} = rL_k'(r) = n$ by our choice of r. Then by (3.7) with $q := \frac{k}{n}$ and $\varrho := nr$ satisfying (3.15) and (3.16), we obtain

$$\upsilon_2(r) = r^2 L_k''(r) + rL_k'(r) = \Big(\frac{24}{\pi^2}(\log 2)^2 - 1 \Big) n + O(1) = \frac{\pi^2}{6}\,\sigma^2 n + O(1).$$

Furthermore, each $\upsilon_j(r) \asymp n$ by (3.7) when $k_- \leqslant k \leqslant k_+$. Thus $\upsilon_j(r)\theta_0^j \to 0$ for $j = 3, 4, \dots$, and we then obtain

$$\begin{aligned} J_I &= \frac{1}{2\pi} \int_{-\theta_0}^{\theta_0} e^{-\frac{1}{2}\upsilon_2(r)\theta^2 - \frac{1}{6}\upsilon_3(r)i\theta^3 + O(n\theta^4)} \,\mathrm{d}\theta \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\upsilon_2(r)\theta^2} \left(1 - \tfrac{1}{6}\upsilon_3(r)i\theta^3 + O(n\theta^4 + n\theta^6)\right) \mathrm{d}\theta + O\big(\upsilon_2^{-1} n^{\frac{3}{8}} e^{-18\upsilon_2(r)n^{-\frac{3}{4}}}\big) \\ &= \frac{1}{\sqrt{2\pi\upsilon_2(r)}}\big(1 + O\big(n^{-1}\big)\big) + O\big(n^{-\frac{5}{8}} e^{-3\pi^2\sigma^2 n^{\frac{1}{4}}}\big), \end{aligned}$$

which proves (3.22). □

Proof of Theorem 5. With (3.21) and (3.22) available, we can now complete the proof of Theorem5 by deriving the finer expansion

$$r^{-n} A_k(r) = c_0 \rho^n n^{n+\frac{1}{2}} e^{-\frac{1}{2}x^2} \left(1 + \frac{g_1(x)}{\sqrt{n}} + O\big(n^{-1}(1 + x^6)\big)\right),$$

where $(c_0, \rho) := \big(\sqrt{\frac{24}{\pi}}, \frac{12}{e\pi^2}\big)$ and $g_1(x)$ is an odd polynomial in x of degree three (whose expression is immaterial here). It follows that

$$\begin{aligned} a_{n,k} := [z^n]A_k(z) &= \frac{\sqrt{3}}{\pi^{3/2}\sigma}\, n^{-\frac{1}{2}} r^{-n} A_k(r)\big(1 + O\big(n^{-1}\big)\big) \\ &= \frac{c_1}{\sigma}\, \rho^n n^n e^{-\frac{1}{2}x^2}\Big(1 + \frac{g_1(x)}{\sqrt{n}} + O\big(n^{-1}(1 + x^6)\big)\Big), \end{aligned} \tag{3.23}$$

uniformly for $k_- \leqslant k \leqslant k_+$, where $(c_1, \rho) := \big(\frac{\sqrt{72}}{\pi^2}, \frac{12}{e\pi^2}\big)$, where σ is given in (3.12).

From this and the two estimates (3.14) and (3.20), we obtain

$$a_n = \frac{12}{\pi^{3/2}}\, \rho^n n^n \sum_{k_- \leqslant k \leqslant k_+} \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}\,\sigma} \left(1 + \frac{g_1(x)}{\sqrt{n}} + O\big(n^{-1}(1 + x^6)\big)\right) + O\big(\rho^n n^{n+\frac{3}{2}} e^{-n^{\frac{1}{4}}}\big),$$

from which we deduce (3.2) by approximating the sum by an integral. □

Remark 1. *We have proved more than the asymptotic estimate (3.2); indeed, if we define the random variable X_n by*

$$\mathbb{P}(X_n = k) := \frac{[z^n]A_k(z)}{[z^n]A(z)} \qquad (1 \leqslant k \leqslant n),$$

then our asymptotic expansions (3.23) and (3.2) imply obviously the local limit theorem (in the form of moderate deviations):

$$\mathbb{P}(X_n = \mu n + x\sigma\sqrt{n}) = \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi\sigma^2 n}} \left(1 + O\Big(\frac{|x| + |x|^3}{\sqrt{n}}\Big)\right),$$

uniformly for $x = o\big(n^{\frac{1}{6}}\big)$.

4. Asymptotic expansions and change of variables

We examine briefly in this section two different ways to obtain asymptotic expansions for $a_n = [z^n]A(z)$ as defined in (3.2), and then show how an argument based on change of variables leads to expansions for the coefficients under different parametrization of the underlying function.

The first approach to derive an expansion of the form

$$[z^n]A(z) = c\rho^n n^{n+\frac{1}{2}}\Bigg(1 + \sum_{1\leqslant j<m} \nu_j n^{-j} + O\big(n^{-m}\big)\Bigg), \tag{4.1}$$

for some computable coefficients ν_j, is now straightforward following the same analysis detailed in the previous section. It consists in first computing an asymptotic expansion for $a_{n,k}$:

$$a_{n,k} = \frac{c_1}{\sigma}\,\rho^n n^n e^{-\frac{1}{2}x^2}\Bigg(1 + \sum_{1\leqslant j<m} \frac{g_j(x)}{n^{\frac{1}{2}j}} + O\big(n^{-\frac{1}{2}m}\big)\Bigg), \quad \text{with} \quad (c,\rho) := \big(\tfrac{12}{\pi^{3/2}}, \tfrac{12}{e\pi^2}\big),$$

which holds uniformly for $k = \mu n + x\sigma\sqrt{n}$, $x = o(n^{\frac{1}{6}})$, where $g_j(x)$ is a computable polynomial in x of degree $3j$ and contains only powers of x with the same parity as j. From this we can then deduce (4.1) by approximating the sum by an integral and extending the integration range to $\pm\infty$. We omit the details as they are more or less standard and all procedures can be readily coded in symbolic computation software.

4.1. **An asymptotic expansion via Dirichlet series.** For more methodological interest, we sketch here another approach, based on that used in [8], to obtain asymptotic expansions for a_n when more information is available.

Proposition 16. *The sequence a_n in (3.2) satisfies the asymptotic expansion*

$$a_n = c\rho^n\, n!\Bigg(1 + \sum_{1\leqslant j<m} \frac{c_j}{n(n-1)\cdots(n-j+1)} + O\big(n^{-m}\big)\Bigg), \tag{4.2}$$

for $m \geqslant 2$, where $(c,\rho) := \big(\frac{6\sqrt{2}}{\pi^2}, \frac{12}{\pi^2}\big)$ and $c_j := \frac{1}{j!}\big(-\frac{\pi^2}{288}\big)^j$ for $j \geqslant 1$.

In particular,

$$a_n = c\rho^n n!\Bigg(1 - \frac{\pi^2}{288n} + \frac{\pi^4}{165888\, n(n-1)} + O\big(n^{-3}\big)\Bigg).$$

The very simple form of the coefficients c_j naturally suggests the following approximation:

$$a_n = c\rho^n n! e^{-\frac{\pi^2}{288\, n}}\left(1 + O\big(n^{-3}\big)\right),$$

which has obvious numerical advantages.

Proof. We begin with (1.7). As in [8], we define the Dirichlet series

$$D(s) := \sum_{n\geqslant 1} n^{-s}[q^{n-1}]R\big(q^{24}\big) = 1 + 25^{-s} - 49^{-s} + 2\cdot 73^{-s} - 2\cdot 97^{-s} + 121^{-s} + \cdots,$$

which converges absolutely for $\Re(s) > 1$ and can be analytically continued into the whole s-plane. Together with Mellin transform techniques (the correspondence between the singular expansion of a function at the origin and the residues of the Mellin transform; see [19, Theorem 3] or [8, Lemma 2.4]), we now have the two relations (see [8])

$$\begin{cases} a_n = \dfrac{(-1)^n}{2}[z^n]R\big(e^{-z}\big), \\ b_n := [z^n]e^{-\frac{1}{24}z}R\big(e^{-z}\big) = \dfrac{(-1)^n D(-n)}{n!24^n}. \end{cases}$$

Then the functional equation (1.6) (derived in [8, 10]) gives

$$D(-n) = c_0\rho_0^n n!^2\left(1+O\big(23^{-n}\big)\right), \quad \text{with} \quad (c_0,\rho_0) := \left(\tfrac{12\sqrt{2}}{\pi^2}, \tfrac{288}{\pi^2}\right),$$

for large n. This implies that

$$b_n = c_0(-1)^n\rho^n n!\left(1+O\big(23^{-n}\big)\right), \quad \text{with} \quad (c_0,\rho) := \left(\tfrac{12\sqrt{2}}{\pi^2}, \tfrac{12}{\pi^2}\right).$$

From this, we have

$$\frac{b_{n-j}}{b_n} = \frac{(-\rho)^{-j}}{n(n-1)\cdots(n-j+1)}\left(1+O\big(23^{-n}\big)\right) \quad (j=0,1,\dots), \tag{4.3}$$

implying that the partial sum

$$a_n = \frac{(-1)^n}{2}\, b_n \sum_{0\leqslant j\leqslant n}\frac{b_{n-j}}{j!24^j b_n}$$

is itself an asymptotic expansion. In this way, we obtain (4.2). □

4.2. **From $[z^n]R(e^z)$ to $[z^n]R(1+z)$ by a change of variables.** We sketch here a different technique to derive the asymptotic expansion (1.4) for $[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big((1+z)^j-1\big)$ from (4.2) for a_n. The original proof by Zagier in [45] and by Bringmann–Li–Rhoades in [8] relies on the asymtotics of the Stirling numbers of the first kind. We give a direct approach via change of variables, which has the advantages of being easily codable and widely applicable in more general contexts; see Sections 5 and 8.

Define $R(q)$ by (1.5). Since

$$R(q) = 2\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(q^j-1\big)$$

is true to infinite order at every root of unity, that is, for any $n \geqslant 1$, the above identity holds for any root of unity $q^n = 1$. In particular, it includes the case $q = 1$ (see [10]). By the change of variables $1+z = e^y$, we have,

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big((1+z)^j-1\big) = \frac{1}{2}[z^n]R(1+z) = [y^n]g(y)\big(e^{\frac{1}{24}y}R\big(e^y\big)\big),$$

where

$$g(y) := \frac{1}{2}\left(\frac{y}{e^y-1}\right)^{n+1}e^{\frac{23}{24}y} = \frac{1}{2}\exp\biggl(-\frac{n}{2}y+\frac{11}{24}y-(n+1)\sum_{j\geqslant 1}\frac{B_{2j}}{2j\cdot(2j)!}\,y^{2j}\biggr),$$

for small y. Since b_n (see (4.3)) grows factorially with n, and the Taylor coefficients of $g(y)$ are small when compared to b_n, we expand g at $y=\eta$, where η is small and to be determined soon, and then carry out term by term extraction of the coefficients, yielding

$$\begin{aligned}[y^n]g(y)\big(e^{\frac{1}{24}y}R\big(e^y\big)\big) &= \sum_{j\geqslant 0}\frac{g^{(j)}(\eta)}{j!}[y^n](y-\eta)^j e^{\frac{1}{24}y}R\big(e^y\big)\\ &= g(\eta)\bar{b}_n + g'(\eta)\big(\bar{b}_{n-1}-\eta\bar{b}_n\big)+\cdots,\end{aligned}$$

where $\bar{b}_n := (-1)^n b_n = [y^n]e^{\frac{1}{24}y}R\big(e^y\big)$. So if we take (see (4.3))

$$\eta := \frac{\bar{b}_{n-1}}{\bar{b}_n} = \frac{\pi^2}{12n}\left(1+O\big(23^{-n}\big)\right),$$

then the terms involving $g'(\eta)$ become zero, and we have

$$[y^n]\big(e^{\frac{1}{24}y}R\big(e^y\big)\big)g(y) = g(\eta)\bar{b}_n\left(1+\frac{g''(\eta)}{2g(\eta)}\Big(\frac{\bar{b}_{n-2}}{\bar{b}_n}-\frac{\bar{b}_{n-1}^2}{\bar{b}_n^2}\Big)+\cdots\right).$$

In general, by estimating the Taylor remainders, we deduce the expansion

$$[y^n]\big(e^{\frac{1}{24}y}R\big(e^y\big)\big)g(y) = g(\eta)\bar{b}_n\left(1+\sum_{2\leqslant j\leqslant 2m}\frac{g^{(j)}(\eta)}{j!g(\eta)}H_j(n)+O\big(n^{-m-1}\big)\right),$$

for $m\geqslant 1$, where the general terms are of order $n^{-\lceil\frac{1}{2}j\rceil}$ because $g^{(j)}(\eta)=O(n^j)$ and

$$H_j(n) := \sum_{0\leqslant \ell\leqslant j}\binom{j}{\ell}\left(-\frac{\pi^2}{12n}\right)^{j-\ell}\frac{\bar{b}_{n-\ell}}{\bar{b}_n} = \left(\frac{\pi^2}{12}\right)^j\sum_{0\leqslant\ell\leqslant j}\binom{j}{\ell}\frac{(-1)^{j-\ell}(n-\ell)!}{n^{j-\ell}n!}\left(1+O\big(23^{-n}\big)\right),$$

which decays in the order $n^{-j-\lceil\frac{1}{2}j\rceil}$. In this way, we obtain

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big((1+z)^j-1\big) = c\rho^n\, n!\left(1+\sum_{1\leqslant j<m}\frac{c_j}{n^j}+O\big(n^{-m}\big)\right), \tag{4.4}$$

where $(c,\rho) := \big(\frac{6\sqrt{2}}{\pi^2}\,e^{-\frac{\pi^2}{24}}, \frac{12}{\pi^2}\big)$ and

$$\begin{aligned}c_1 &:= \frac{\pi^2(\pi^2+66)}{1728}\approx 0.43333, \quad c_2 := \frac{\pi^4(\pi^4-12\pi^2-3420)}{5971968}\approx -0.05612,\\ c_3 &:= -\frac{\pi^4(95\pi^8+9360\pi^6-232416\pi^4-27051840\pi^2+709171200)}{1238347284480}\approx -0.03378.\end{aligned}$$

5. A framework for matrices with 1s

We consider in this section generating functions of the form

$$\sum_{k\geqslant 0}d(z)^{k+\omega_0}\prod_{1\leqslant j\leqslant k}\big(e(z)^{j+\omega}-1\big)^{\alpha}, \tag{5.1}$$

for $\alpha\in\mathbb{Z}^+$ and $\omega_0,\omega\in\mathbb{C}$, where $d(z)$ and $e(z)$ are functions analytic at $z=0$ and satisfies $d(0)>0$, $e(0)=1$ and $e'(0)\neq 0$. Then we discuss applications to row-Fishburn and Fishburn matrices with entry restrictions and some OEIS sequences.

Our approach consists in examining first the asymptotics of the simpler pattern

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(e^{(j+\omega)z}-1\right)^{\alpha},$$

for $\alpha \in \mathbb{Z}^+$ and $\omega \in \mathbb{C}$, and follows closely the detailed analysis given in Section 3 for the sequence A158690. Then the extension to (5.1) will rely on the change of variables argument of Section 4.2.

Proposition 17. *For any $\alpha \in \mathbb{Z}^+$ and $\omega \in \mathbb{C}$,*

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(e^{(j+\omega)z}-1\right)^{\alpha} \simeq c\rho^n n^{n+\alpha\omega+\frac{1}{2}\alpha}, \tag{5.2}$$

uniformly in ω, where the notation "$\simeq$" is defined in (1.3) and

$$(c,\rho) := \left(\frac{\sqrt{6}}{\alpha\pi}\left(\frac{2\sqrt{6}}{\sqrt{\alpha\pi}\,\Gamma(1+\omega)}\left(\frac{12}{\alpha\pi^2}\right)^{\omega}\right)^{\alpha}, \frac{12}{e\alpha\pi^2}\right).$$

When $\omega \in \mathbb{Z}^-$, the leading constant c is interpreted as zero because of $\Gamma(1+\omega)$ in the denominator, and the right-hand side of (5.2) becomes then a big-O estimate.

Proof. We sketch the major steps for obtaining the dominant term, as the error term follows from the same procedure with more refined calculations.

- By the Euler-Maclaurin formula (3.3) (with $I(z)$ defined in (3.5))

$$\begin{aligned}\sum_{1\leqslant j\leqslant k}\log\left(e^{(j+\omega)z}-1\right) &= k\log\left(e^{kz}-1\right) - \frac{I(kz)}{z} + \left(\omega+\frac{1}{2}\right)\log\frac{e^{kz}-1}{z}\\ &\qquad - \log\Gamma(1+\omega) + \frac{\log 2\pi}{2} + O\left(|\omega|^2\left(k^{-1}+|z|\right)\right),\end{aligned} \tag{5.3}$$

  (compare (3.4)) which holds uniformly as $k \to \infty$ and $k|z| \leqslant 2\pi - \varepsilon$ in the sector $|\arg z| \leqslant \pi - \varepsilon$. Here (5.3) holds when $\omega \neq \mathbb{R}^-$. But the asymptotic approximation, by taking the exponential on both sides of (5.3),

$$\begin{aligned}&\prod_{1\leqslant j\leqslant k}\left(e^{(j+\omega)z}-1\right)\\ &\qquad = \frac{\sqrt{2\pi}}{\Gamma(1+\omega)}\left(\frac{e^{kz}-1}{z}\right)^{\omega+\frac{1}{2}}\left(e^{kz}-1\right)^k e^{-I(kz)/z}\left(1+O\left(|\omega|^2\left(k^{-1}+|z|\right)\right)\right)\end{aligned}$$

  does hold for bounded ω, provided we interpret the factor $\frac{1}{\Gamma(1+\omega)}$ as zero when $\omega \in \mathbb{Z}^-$.
- The saddle-point equation satisfies asymptotically, by the same differentiation argument used for deriving (3.6),

$$\frac{\alpha}{r}I(kr) + \frac{\alpha}{2}(2\omega+1)\left(\frac{kr}{1-e^{-kr}}-1\right) + O\left(k^{-1}+r\right) = n.$$

  Since the dominant term is independent of ω, we deduce that $k = qn$ with $q \sim \frac{\mu}{\alpha}$ and $rn \sim \alpha\xi$, where $(\mu,\xi) := \left(\frac{12}{\pi^2}\log 2, \frac{\pi^2}{12}\right)$ is the same as in (3.11).

- Observe that for large $k \leqslant n$

$$\prod_{1\leqslant j\leqslant k}\left|e^{(j+\omega)z}-1\right| = O\big(k^{\Re(\omega)}\big)\prod_{1\leqslant j\leqslant k}\left|e^{jz}-1\right|,$$

when $|z| \asymp n^{-1}$ and $\omega = O(1)$. Then the smallness of the sum

$$\sum_{|k-\frac{\mu}{\alpha}n|\geqslant\sqrt{2}\sigma n^{\frac{5}{8}}}[z^n]\prod_{1\leqslant j\leqslant k}\big(e^{(j+\omega)z}-1\big)^{\alpha},$$

as well as the corresponding sum of integrals $\sum_{|k-\frac{\mu}{\alpha}n|\leqslant\sqrt{2}\sigma n^{\frac{5}{8}}}\int_{6n^{-\frac{3}{8}}\leqslant|\theta|\leqslant\pi}$ follows from the same bounding techniques used in the proofs of Propositions 12 and 14.

- Inside the central range $\frac{1}{\alpha}k_- \leqslant k \leqslant \frac{1}{\alpha}k_+$, where $k_\pm := \mu n \pm \sqrt{2}\,\sigma n^{\frac{5}{8}}$, write, as before, $q := \frac{1}{\alpha}\big(\mu + \sigma\frac{x}{\sqrt{n}}\big)$, and solve the saddle-point equation for r, giving

$$rn = \alpha\xi + \frac{\alpha\xi_1 x}{\sqrt{n}} + \frac{\alpha^2\xi_2(1+2\omega)+\alpha\xi_3x^2}{n} + O\Big(\frac{|x|+|x|^3}{n^{3/2}}\Big), \tag{5.4}$$

where ξ_i are defined in (3.17).

- We then obtain

$$r^{-n}\prod_{1\leqslant j\leqslant k}\big(e^{(j+\omega)r}-1\big)^{\alpha} \sim c_0\rho^n n^{n+\alpha(\frac{1}{2}+\omega)},$$

where

$$(c_0,\rho) := \left(\left(\frac{2\sqrt{6}}{\sqrt{\alpha\pi}\,\Gamma(1+\omega)}\Big(\frac{12}{\alpha\pi^2}\Big)^{\omega}\right)^{\alpha}, \frac{12}{e\alpha\pi^2}\right).$$

- The remaining saddle-point analysis is similar to that of Theorem 5.

□

The uniformity in ω will be needed in Section 7. We now consider the framework (5.1).

Theorem 18. *Assume $\alpha \in \mathbb{Z}^+$ and $\omega_0, \omega \in \mathbb{C}$. For any two functions $d(z)$ and $e(z)$ that are analytic at $z = 0$, satisfying $d(0) = e(0) = 1$ and $e'(0) \neq 0$, we have*

$$[z^n]\sum_{k\geqslant 0} d(z)^{k+\omega_0}\prod_{1\leqslant j\leqslant k}\big(e(z)^{j+\omega}-1\big)^{\alpha} \simeq c\rho^n n^{n+\alpha(\frac{1}{2}+\omega)}, \tag{5.5}$$

uniformly for bounded ω_0 and ω, where $d_j := [z^j]d(z)$, $e_j := [z^j]e(z)$, and

$$(c,\rho) := \left(\frac{\sqrt{6}}{\alpha\pi}\left(\frac{2\sqrt{6}}{\sqrt{\alpha\pi}\,\Gamma(1+\omega)}\Big(\frac{12}{\alpha\pi^2}\Big)^{\omega}\right)^{\alpha} 2^{\frac{d_1}{e_1}} e^{\frac{\alpha\pi^2}{12}\left(\frac{e_2}{e_1^2}-\frac{1}{2}\right)}, \frac{12e_1}{e\alpha\pi^2}\right). \tag{5.6}$$

The situation when $d(0) \neq 1$ is readily modified. Also the error term can be further refined if needed. The case when $e'(0) = 0$ but $e''(0) > 0$ will be treated in Section 8 with particular applications to self-dual Fishburn matrices.

We see that the exponential term depends on α and e_1, the polynomial term on α and ω, and the leading constant c on α, ω, d_1, e_1 and e_2. Furthermore, as far as the dominant asymptotics of the coefficients is concerned, the difference in (5.5) and (5.2) is reflected exclusively via the first three terms d_1, e_1, e_2 in the Taylor expansions of $d(z)$ and $e(z)$.

Proof. By Cauchy's integral formula

$$a_n := \frac{1}{2\pi i}\oint_{|z|=r_0} z^{-n-1}\sum_{1\leqslant k\leqslant \frac{n}{\alpha}} d(z)^{k+\omega_0}\prod_{1\leqslant j\leqslant k}\big(e(z)^{j+\omega}-1\big)^{\alpha}\,\mathrm{d}z,$$

where $r_0>0$. Since $e(z)=1+e_1z+\cdots$ with $e_1\neq 0$, the function is locally invertible and we can make the change of variables $e(z)=e^y$, giving

$$a_n = \frac{1}{2\pi i}\oint_{|y|=r}\psi'(y)\psi(y)^{-n-1}\sum_{1\leqslant k\leqslant \frac{n}{\alpha}} d(\psi(y))^{k+\omega_0}A_k(y)\,\mathrm{d}y,$$

where $A_k(y):=\prod_{1\leqslant j\leqslant k}\big(e^{(j+\omega)y}-1\big)^{\alpha}$ and $\psi(y)$ satisfies $\psi(0)=0$ and $e(\psi(y))=e^y$. By the analyticity of $e(z)$ at the origin, $\psi(y)$ is also analytic at $y=0$. In particular,

$$\psi_1=[y]\psi(y)=\tfrac{1}{e_1}\quad\text{and}\quad \psi_2=[y^2]\psi(y)=\tfrac{1}{e_1}\big(\tfrac12-\tfrac{e_2}{e_1^2}\big). \tag{5.7}$$

By the analyticity of d and ψ at the origin, we have, for small $|y|$,

$$d(\psi(y))^{k+\omega_0}=\big(1+d_1\psi_1y+\big(d_1\psi_2+d_2\psi_1^2\big)y^2+\cdots\big)^k;$$

on the other hand, from our saddle-point analysis above, the integration path $|y|=r$ is very close to zero with $r\asymp n^{-1}$, and most contribution to a_n comes from terms with k of linear order, so we see that $d(\psi(y))^k$ is bounded and close to $e^{d_1\psi_1ky}$ for large n. Similarly, by (5.7),

$$\begin{aligned}\psi'(y)\psi(y)^{-n-1}&=\big(\psi_1+2\psi_2y+O\big(|y|^2\big)\big)\big(\psi_1y+\psi_2y^2+O\big(|y|^3\big)\big)^{-n-1}\\&=e_1^ny^{-n-1}e^{-\frac{\psi_2}{\psi_1}ny}\big(1+O\big(|y|+n|y|^2\big)\big).\end{aligned}$$

Thus the same proof of Theorem 5 extends *mutatis mutandis* to this case, and we then obtain the asymptotic approximation

$$\begin{aligned}a_n=&\sum_{\frac{k_-}{\alpha}\leqslant k\leqslant\frac{k_+}{\alpha}}\frac{1}{2\pi i}\oint_{|y|=r}y^{-n-1}A_k(y)e^{-\frac{\psi_2}{\psi_1}ny+d_1\psi_1ky}\big(1+O\big(|y|+n|y|^2\big)\big)\,\mathrm{d}y\\&+O\big(\rho^nn^{n+\alpha(\Re(\omega)+\frac12)}e^{-n^{\frac14}}\big),\end{aligned}$$

where $k_\pm:=\mu n\pm\sqrt{2}\,\sigma n^{\frac58}$ and r satisfies (5.4). Since $q=\frac{k}{n}$ satisfies $q=\frac{1}{\alpha}\big(\mu+\sigma\frac{x}{\sqrt{n}}\big)$, we then deduce (5.5) by noting that

$$e^{-\frac{\psi_2}{\psi_1}nr+d_1\psi_1kr}=e^{-\frac{\psi_2}{\psi_1}\alpha\xi+d_1\psi_1\mu\xi}\Big(1+\frac{\tilde g_1(x)}{\sqrt{n}}+\frac{\tilde g_2(x)}{n}+\cdots\Big),$$

for some polynomials $\tilde g_1(x)$ and $\tilde g_1(x)$, where (μ,ξ) is given in (3.11). □

6. Applications I. Univariate asymptotics

We group in this section various examples (mostly from the OEIS) according to the pair (α,ω). Some of them were already analyzed in the OEIS by Kotěšovec, but without proofs.

6.1. Λ-row-Fishburn matrices and examples with $(\alpha,\omega):=(1,0)$. We derive a general asymptotic approximation to the number of Λ-row-Fishburn matrices and discuss some other examples.

6.1.1. Λ-*row-Fishburn matrices.* From Theorem 18, it is clear that no matter how widely we choose the nonnegative integers as entries, the number of the resulting row-Fishburn matrices of size n depends only on the numbers of 1s and 2s as far as the leading order asymptotics is concerned, provided that the generating function satisfies (6.1) and is analytic at $z = 0$.

Corollary 19. *Let Λ be a multiset of nonnegative integers with the generating function*

$$\Lambda(z) := 1 + \sum_{\lambda\in\Lambda} z^{\lambda} = 1 + \lambda_1 z + \lambda_2 z^2 + \cdots . \tag{6.1}$$

If $\Lambda(z)$ is analytic at $z = 0$ and $\Lambda'(0) = \lambda_1 > 0$, then the number of Λ-row-Fishburn matrices of size n satisfies

$$[z^n] \sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \left(\Lambda(z)^j - 1\right) \simeq c\rho^n n^{n+\frac{1}{2}} \quad \textit{with} \quad (c, \rho) := \left(\tfrac{12}{\pi^{3/2}}\, e^{\frac{\pi^2}{12}\left(\frac{\lambda_2}{\lambda_1^2}-\frac{1}{2}\right)}, \tfrac{12\,\lambda_1}{e\pi^2}\right). \tag{6.2}$$

Proof. Apply Theorem 18 with $(d(z), e(z)) := (1, \Lambda(z))$. ☐

In particular, this corollary applies to the OEIS sequences in Table 3.

OEIS	Λ	$\Lambda(z)$	(λ_1, λ_2)	c	ρ
A179525	$\{0, 1\}$	$1 + z$	$(1, 0)$	$\frac{12}{\pi^{3/2}}\, e^{-\frac{\pi^2}{24}}$	$\frac{12}{e\pi^2}$
A289316	$\{0\} \cup \{2k - 1 : k \in \mathbb{Z}^+\}$	$\frac{1+z-z^2}{1-z^2}$	$(1, 0)$	$\frac{12}{\pi^{3/2}}\, e^{-\frac{\pi^2}{24}}$	$\frac{12}{e\pi^2}$
A207433	$\{0, 1, 2\}$	$\frac{1-z^3}{1-z}$	$(1, 1)$	$\frac{12}{\pi^{3/2}}\, e^{\frac{\pi^2}{24}}$	$\frac{12}{e\pi^2}$
A158691	$\mathbb{Z}_{\geqslant 0}$	$\frac{1}{1-z}$	$(1, 1)$	$\frac{12}{\pi^{3/2}}\, e^{\frac{\pi^2}{24}}$	$\frac{12}{e\pi^2}$
A289313	$\{0, 1, 1, 2, 2, \ldots\}$	$\frac{1+z}{1-z}$	$(2, 2)$	$\frac{12}{\pi^{3/2}}$	$\frac{24}{e\pi^2}$

Table 3. The large-n asymptotics (6.2) of some OEIS sequences that correspond to the enumeration of Λ-row-Fishburn matrices with different Λ. Here we split the pair (c, ρ) for clarity and group the sequences with the same pair (λ_1, λ_2).

The last sequence of Table 3 can also be interpreted as the number of upper triangular matrices with integer entries (positive and negative) whose sum of absolute entries is n, and no row sums (in absolute entries) to zero.

6.1.2. *Some OEIS sequences.* Some other OEIS examples with $(\alpha, \omega) := (1, 0)$ are compiled in Table 4, where they all satisfy the asymptotic pattern

$$[z^n] \sum_{k\geqslant 0} d(z)^k \prod_{1\leqslant j\leqslant k} \left(e(z)^j - 1\right) \simeq c\rho^n n^{n+\frac{1}{2}}. \tag{6.3}$$

Note that the Taylor expansions of $e(z)$ in the two cases A207386 and A207397 of Table 4 both contain negative coefficients.

OEIS	$d(z)$	$e(z)$	(d_1, e_1, e_2)	(c, ρ)
A158690	1	e^z	$(0, 1, \frac{1}{2})$	$\left(\frac{12}{\pi^{3/2}}, \frac{12}{e\pi^2}\right)$
A196194	$\frac{z}{e^z-1}$	e^z	$\left(-\frac{1}{2}, 1, \frac{1}{2}\right)$	$\left(\frac{6\sqrt{2}}{\pi^{3/2}}, \frac{12}{e\pi^2}\right)$
A207214	e^z	e^z	$(1, 1, \frac{1}{2})$	$\left(\frac{24}{\pi^{3/2}}, \frac{12}{e\pi^2}\right)$
A207386	1	$\frac{1+z}{1+z^3}$	$(0, 1, 0)$	$\left(\frac{12}{\pi^{3/2}}\, e^{-\frac{\pi^2}{24}}, \frac{12}{e\pi^2}\right)$
A207397	1	$\frac{1+z}{1+z^2}$	$(0, 1, -1)$	$\left(\frac{12}{\pi^{3/2}}\, e^{-\frac{\pi^2}{8}}, \frac{12}{e\pi^2}\right)$
A207556	$1+z$	$1+z$	$(1, 1, 0)$	$\left(\frac{24}{\pi^{3/2}}\, e^{-\frac{\pi^2}{24}}, \frac{12}{e\pi^2}\right)$

Table 4. Some OEIS examples with $(\alpha, \omega) := (1, 0)$; they all satisfy the asymptotic pattern (6.3) with (c, ρ) given in the last column. All ρ's are the same because $e'(0) = e_1 = 1$.

6.1.3. *Minor variants.* Consider the following sequence (A207652) whose generating function does not have the same pattern (5.1); yet this sequence has the same leading order asymptotics as A179525 (see Table 3):

$$[z^n] \sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \frac{(1+z)^j - 1}{1 - z^j} \simeq c\rho^n n^{n+\frac{1}{2}}, \quad \text{with} \quad (c, \rho) := \left(\tfrac{12}{\pi^{3/2}} e^{-\frac{\pi^2}{24}}, \tfrac{12}{e\pi^2}\right).$$

This is because the extra product

$$\prod_{1\leqslant j\leqslant k} \frac{1}{1 - z^j} = 1 + z + O(|z|^2) \tag{6.4}$$

is asymptotically unity plus a negligible error when $z \asymp n^{-1}$. Similarly, the sequence A207653 satisfies

$$[z^n] \sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \frac{1 - (1-z)^{2j-1}}{1 - z^{2j-1}} \simeq c\rho^n n^{n+\frac{1}{2}}, \quad \text{with} \quad (c, \rho) := \left(\tfrac{12}{\pi^{3/2}} e^{\frac{\pi^2}{24}}, \tfrac{12}{e\pi^2}\right), \tag{6.5}$$

which has the same leading-order asymptotics as A158691.

Another example is A207434, which is defined by

$$b_n := n[z^n] \log\Bigg(\sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \left((1+z)^j - 1\right) \Bigg).$$

This is not of our format (5.1) but the leading asymptotics can be quickly linked to that of A179525, the number a_n of primitive row-Fishburn matrices of size n; see (1.4). By the relation

$$b_n = n a_n - \sum_{1\leqslant j<n} b_j a_{n-j} \qquad (n \geqslant 1),$$

and the factorial growth of the coefficients in (1.4), we then deduce that

$$b_n \simeq n a_n \simeq c\rho^n n^{n+\frac{3}{2}}, \quad \text{with} \quad (c, \rho) := \left(\tfrac{12}{\pi^{3/2}}\, e^{-\frac{\pi^2}{24}}, \tfrac{12}{e\pi^2}\right).$$

6.1.4. *Recursive variants.* Consider first the sequence A186737 whose generating function is defined recursively by

$$f(z) = \sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \big((1+zf(z))^j - 1\big) = 1 + z + 3z^2 + 14z^3 + 82z^4 + 563z^5 + \cdots . \tag{6.6}$$

Let $f_m = [z^m]f(z)$. Then by connecting (6.6) to row-Fishburn matrices, we find that f_n is the sum of weights on row-Fishburn matrices of size n, where each matrix is assigned a weight equal to the product of f_{j-1} for each entry $j > 0$.

This is close to the framework (5.1). While Theorem 18 does not apply, the proof there does. More precisely, we first define $f_n(z) := \sum_{0\leqslant j\leqslant n} a_n z^n$, where $a_n := [z^n]f(z)$, so that

$$a_n = [z^n] \sum_{0\leqslant k\leqslant n} \prod_{1\leqslant j\leqslant k} \big((1+zf_n(z))^j - 1\big).$$

Before performing the change of variables $e^y = f_n(z)$ for $n \geqslant 1$, we need to prove that $|f_n(z)|$ remains bounded when $|z| \asymp n^{-1}$. We begin by the trivial bounds, using the positivity of a_n:

$$\begin{aligned} a_n &\leqslant r^{-n} \sum_{1\leqslant k\leqslant n} \prod_{1\leqslant j\leqslant k} \big((1+rf_n(r))^j - 1\big) \\ &\leqslant r^{-n} \sum_{1\leqslant k\leqslant n} (1+rf_n(r))^{\binom{k+1}{2}} \\ &\leqslant nr^{-n}(1+rf_n(r))^{\binom{n+1}{2}}. \end{aligned}$$

Here $r = r_n > 0$ is chosen to be the solution of the equation

$$\frac{n}{\binom{n+1}{2}} = \frac{2}{n+1} = r\,\partial_r \log(1+rf_n(r)) = r\frac{f_n(r)+rf_n'(r)}{1+rf_n(r)}. \tag{6.7}$$

Now, by the monotonicity of a_n, we obtain

$$r\frac{f_n(r)+rf_n'(r)}{1+rf_n(r)} \sim \begin{cases} \dfrac{\sum_{1\leqslant j\leqslant n+1} ja_{j-1}r^j}{1+\sum_{1\leqslant j\leqslant n+1} a_{j-1}r^j} \sim n, & \text{if } r\to\infty; \\ \dfrac{r+O(r^2)}{1+O(r)} = r + O(r^2), & \text{if } r \to 0. \end{cases}$$

Thus there exists a unique saddle-point $r > 0$ solving the equation (6.7). Since $2/(n+1) \to 0$, such r satisfies $r \to 0$.

On the other hand, the inequality $rf_n'(r) \geqslant f_n(r) - 1$ leads to

$$\frac{2}{n+1} = r\frac{f_n(r)+rf_n'(r)}{1+rf_n(r)} \geqslant \frac{2rf_n(r)-r}{1+rf_n(r)},$$

which in turn implies that

$$f_n(r) \leqslant \frac{1}{rn} + \frac{1}{2} + \frac{1}{2n}.$$

Thus $f_n(r) = O(1)$ when $r \asymp n^{-1}$. Now, a direct bootstrapping argument based on (6.7) gives the finer expansion

$$r = \frac{2}{n} - \frac{6}{n^2} - \frac{30}{n^3} + \cdots ,$$

and then

$$f_n(r) = 1 + \frac{2}{n} + \frac{6}{n^2} + \cdots .$$

Consequently,

$$a_n \leqslant nr^{-n}(1 + rf_n(r))^{\binom{n+1}{2}} = O\big(n^{n+1}\rho^n\big), \quad \text{with} \quad \rho := \frac{e}{2}.$$

Since $|f_n(z)| \leqslant f_n(|z|) = O(1)$ when $z \asymp n^{-1}$, the change of variables $1 + zf_n(z) = e^y$ is not only locally invertible but also leads to the boundedness of the positive solution $z = z(y)$ solving $1 + zf_n(z) = e^y$ when $y = O\big(n^{-1}\big)$. As a result, the local expansion of the solution is given by

$$z = y - \tfrac{1}{2}y^2 - \tfrac{11}{6}y^3 - \tfrac{145}{24}y^4 + \cdots .$$

The remaining analysis then follows the same procedure as the proof of Theorem 18, yielding

$$[z^n] \sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} ((1 + zf(z))^j - 1) \simeq c\rho^n n^{n+\frac{1}{2}}, \quad \text{with} \quad (c, \rho) := \big(\tfrac{12}{\pi^{3/2}} e^{\frac{\pi^2}{24}}, \tfrac{12}{e\pi^2}\big),$$

which is consistent with the expression derived by Kotěšovec on the OEIS page.

Similarly, the sequence A224885 defined as the coefficients of the generating function

$$f(z) = 1 + z + \sum_{k\geqslant 2} \prod_{1\leqslant j\leqslant k} \big(f(z)^j - 1\big) = 1 + z + 2z^2 + 15z^3 + 143z^4 + 1552z^5 + \cdots$$

satisfies

$$[z^n]f(z) \simeq c\rho^n n^{n+\frac{1}{2}}, \quad \text{with} \quad (c, \rho) = \big(\tfrac{12}{\pi^{3/2}}\, e^{\frac{\pi^2}{8}}, \tfrac{12}{e\pi^2}\big).$$

In analogy to (6.6), here $f_n = [z^n]f(z)$ is the sum of weights on row-Fishburn matrices of size n, where the only row-Fishburn matrix of dimension one is primitive and each row-Fishburn matrix is assigned a weight equal to the product of f_j for each entry $j > 0$.

6.2. **Λ-Fishburn matrices and examples with** $(\alpha, \omega) := (2, 0)$**.** We now consider the case when $(\alpha, \omega) := (2, 0)$, beginning with the asymptotics of Λ-Fishburn matrices.

6.2.1. Λ-*Fishburn matrices.*

Corollary 20. *Let* Λ *be a multiset of nonnegative integers with the generating function* $\Lambda(z)$ *defined as in* (6.1)*. If* $\Lambda(z)$ *is analytic at* $z = 0$ *and* $\lambda_1 > 0$*, then the number of Fishburn matrices of size* n *satisfies*

$$[z^n] \sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \big(1 - \Lambda(z)^{-j}\big) \simeq c\rho^n n^{n+1} \quad \textit{with} \quad (c, \rho) := \Big(\tfrac{12\sqrt{6}}{\pi^2} e^{\frac{\pi^2}{6}\left(\frac{\lambda_2}{\lambda_1^2} - \frac{1}{2}\right)}, \tfrac{6\,\lambda_1}{e\pi^2}\Big).$$

Proof. Use (2.5) and then apply Theorem 18 with $d(z) = e(z) = \Lambda(z)$ and $\alpha = 2$. □

A few OEIS examples to which this corollary applies are collected in Table 5.

In particular, we see from Table 5 that Zagier's result (1.1) for the asymptotics of Fishburn numbers corresponds to A022493. Also the result for A138265 improves the crude bound given in [33]; see also [7].

OEIS	Λ	$\Lambda(z)$	(λ_1, λ_2)	(c, ρ)
A022493	$\mathbb{Z}_{\geqslant 0}$	$\frac{1}{1-z}$	$(1, 1)$	$\left(\frac{12\sqrt{6}}{\pi^2}\, e^{\frac{\pi^2}{12}}, \frac{6}{e\pi^2}\right)$
A138265	$\{0, 1\}$	$1+z$	$(1, 0)$	$\left(\frac{12\sqrt{6}}{\pi^2}\, e^{-\frac{\pi^2}{12}}, \frac{6}{e\pi^2}\right)$
A289317	$\{0\} \cup \{2k-1 : k \in \mathbb{Z}^+\}$	$\frac{1+z-z^2}{1-z^2}$	$(1, 0)$	$\left(\frac{12\sqrt{6}}{\pi^2}\, e^{-\frac{\pi^2}{12}}, \frac{6}{e\pi^2}\right)$
A289312	$\{0\} \cup 2\mathbb{Z}^+$	$\frac{1+z}{1-z}$	$(2, 2)$	$\left(\frac{12\sqrt{6}}{\pi^2}, \frac{12}{e\pi^2}\right)$

Table 5. The large-n asymptotics (of the form $c\rho^n n^{n+1}$) of some OEIS sequences that correspond to the enumeration of Λ-Fishburn matrices with different Λ.

Corollary 20 also implies the asymptotics of r-Fishburn numbers [22]:

$$[z^n] \sum_{k \geqslant 0} \prod_{1 \leqslant j \leqslant k} \left(1 - (1-z)^{rj}\right) \simeq c\rho^n n^{n+1} \quad \text{with} \quad (c, \rho) := \left(\frac{12\sqrt{6}}{\pi^2}\, e^{\frac{\pi^2}{12r}}, \frac{6r}{e\pi^2}\right),$$

and applies to the sequence studied in [13] with $\Lambda(z) = 1 + z + \cdots + z^{m-1}$, $m \geqslant 3$.

6.2.2. *Other OEIS examples.* We discuss three other OEIS sequences with $(\alpha, \omega) = (2, 0)$. Consider first A079144, which enumerates labelled interval orders on n points [7] with $d(z) = e(z) = e^z$, and we obtain

$$\begin{aligned}[z^n] \sum_{k \geqslant 0} \prod_{1 \leqslant j \leqslant k} \left(1 - e^{-jz}\right) &= [z^n] \sum_{k \geqslant 0} e^{(k+1)z} \prod_{1 \leqslant j \leqslant k} \left(e^{jz} - 1\right)^2 \\ &\simeq c\rho^n n^{n+1}, \quad \text{with} \quad (c, \rho) := \left(\tfrac{12\sqrt{6}}{\pi^2}, \tfrac{6}{e\pi^2}\right).\end{aligned}$$

Alternatively, (1.2) provides a different proof for this asymptotic estimate and a finer expansion; see [45].

Consider now A207651, the generating function of this sequence is different from A022493, the Fishburn numbers, but they satisfy the same asymptotic relation (see (1.1))

$$[z^n] \sum_{k \geqslant 0} \prod_{1 \leqslant j \leqslant k} \frac{1 - (1-z)^j}{1 - z^j} \simeq c\rho^n n^{n+1}, \quad \text{with} \quad (c, \rho) := \left(\tfrac{12\sqrt{6}}{\pi^2} e^{\frac{\pi^2}{12}}, \tfrac{6}{e\pi^2}\right),$$

since the additional product is again asymptotically 1 plus a negligible error; see (6.4).

The last sequence is A035378:

$$[z^n] \sum_{k \geqslant 1} \prod_{1 \leqslant j \leqslant k} \left(1 - (z-1)^j\right) = [z^n] \sum_{k \geqslant 0} (z-1)^{-k-1} \prod_{1 \leqslant j \leqslant k} \left(1 - (z-1)^{-j}\right)^2.$$

Theorem 18 does not apply directly but our approach does by rewriting the GF as (by grouping the terms in pairs)

$$\sum_{k \geqslant 0} \frac{1}{(1-z)^{2k+1}} \left(\frac{1}{1-z}\left(1 + \frac{1}{(1-z)^{2k+1}}\right)^2 - 1\right) \prod_{1 \leqslant j \leqslant 2k} \left(\frac{1}{(1-z)^j} - 1\right)^2;$$

we then derive the approximation

$$[z^n] \sum_{k \geqslant 1} \prod_{1 \leqslant j \leqslant k} \left(1 - (z-1)^j\right) \simeq c\rho^n n^{n+1}, \quad \text{with} \quad (c, \rho) := \left(\tfrac{48\sqrt{3}}{\pi^2}\, e^{\frac{\pi^2}{48}}, \tfrac{24}{e\pi^2}\right),$$

consistent with that provided on the OEIS webpage of A035378 by Kotěšovec; see also [45, Sec. 5].

6.3. **Examples with $\omega \neq 0$.** We gather some examples in the following table, where we use the form

$$a_n := [z^n] \sum_{k\geqslant 0} d_k(z) \prod_{1\leqslant j\leqslant k} \big(e_j(z)-1\big),$$

with $(d_k(z), e_j(z))$ given in the second column.

OEIS	$(d_k(z), e_j(z))$	$a_n \simeq$	(c,ρ)
A215066	$(1, e^{(2j-1)z})$	$c\rho^n n^n$	$\big(\frac{2\sqrt{3}}{\pi}, \frac{24}{e\pi^2}\big)$
A209832	$(e^{(k+1)z}, e^{(2j-1)z})$	$c\rho^n n^n$	$\big(\frac{2\sqrt{6}}{\pi}, \frac{24}{e\pi^2}\big)$
A214687	$\big(e^{2kz}, e^{(2j-1)z}\big)$	$c\rho^n n^n$	$\big(\frac{4\sqrt{3}}{\pi}, \frac{24}{e\pi^2}\big)$
A207569	$\big(1, (1+z)^{2j-1}\big)$	$c\rho^n n^n$	$\big(\frac{2\sqrt{3}}{\pi}e^{-\frac{\pi^2}{48}}, \frac{24}{e\pi^2}\big)$
A207570	$\big(1, (1+z)^{3j-2}\big)$	$c\rho^n n^{n-\frac{1}{6}}$	$\left(\frac{\Gamma(\frac{2}{3})3^{5/6}}{2^{1/3}\pi^{7/6}}e^{-\frac{\pi^2}{72}}, \frac{36}{e\pi^2}\right)$
A207571	$\big(1, (1+z)^{3j-1}\big)$	$c\rho^n n^{n+\frac{1}{6}}$	$\left(\frac{12^{2/3}}{\pi^{5/6}\Gamma(\frac{2}{3})}e^{-\frac{\pi^2}{72}}, \frac{36}{e\pi^2}\right)$

In general,

$$[z^n] \sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \big((1+z)^{pj-s}-1\big) \simeq c\rho^n n^{n+\frac{1}{2}-\frac{s}{p}},$$

for $0 < s < p$ (not necessarily integers), where

$$(c,\rho) := \left(\frac{\sqrt{\pi}}{\Gamma\big(1-\frac{s}{p}\big)}\left(\frac{\pi^2}{12}\right)^{\frac{s}{p}-1} e^{-\frac{\pi^2}{24p}}, \frac{12p}{e\pi^2}\right).$$

A minor variant of A207569 with the same asymptotic approximation is the sequence A207654:

$$[z^n] \sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \frac{(1+z)^{2j-1}-1}{1-z^{2j-1}} \simeq c\rho^n n^n, \quad \text{with} \quad (c,\rho) = \big(\tfrac{2\sqrt{3}}{\pi}e^{-\frac{\pi^2}{48}}, \tfrac{24}{e\pi^2}\big);$$

see also (6.5).

The last example is A207557: $[z^n]f(z)$ with

$$f(z) := \sum_{k\geqslant 0} (1+z)^{-k(k-1)} \prod_{1\leqslant j\leqslant k} \big((1+z)^{2j-1}-1\big),$$

which can be transformed, by the Rogers-Fine identity (see [15]) into

$$f(z) = 1 + z^{-1} \sum_{k\geqslant 1} (1+z)^{2k+1} \prod_{1\leqslant j\leqslant k} \big((1+z)^{2j-1}-1\big)^2.$$

We can then apply Theorem 18, and obtain

$$\begin{aligned}[z^n]f(z) &= [z^{n+1}] \sum_{k\geqslant 1} (1+z)^{2k+1} \prod_{1\leqslant j\leqslant k} \big((1+z)^{2j-1}-1\big)^2 \\ &\simeq c_0\rho^{n+1}(n+1)^{n+1} \simeq c_0 e\rho\rho^n n^{n+1}, \quad \text{with} \quad (c_0,\rho) := \big(\tfrac{2\sqrt{6}}{\pi}e^{-\frac{\pi^2}{24}}, \tfrac{12}{e\pi^2}\big).\end{aligned}$$

Thus $c_0 e\rho = \frac{24\sqrt{6}}{\pi^3} e^{-\frac{\pi^2}{24}}$, consistent with the expression derived by Kotěšovec in the OEIS; see [37, A207557].

7. Applications II. Bivariate asymptotics (asymptotic distributions)

We derive in this section the various limit laws arising from the sizes of the first row and the diagonal, as well as the number of 1s in random Fishburn and row-Fishburn matrices, assuming that all matrices of the same size are equally likely to be selected. We begin with row-Fishburn matrices because they are technically simpler.

7.1. **Statistics on Λ-row-Fishburn matrices.**

By Proposition 1, the number of Λ-row-Fishburn matrices of size n is given by (see (2.4))

$$a_n := [z^n] \sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \bigl(\Lambda(z)^j - 1\bigr),$$

where $\Lambda(z)$ is the generating function of the multiset Λ; see (2.3). The asymptotics of a_n is already examined in Corollary 19.

Recall that the probability generating function of a Poisson distribution with mean $\tau > 0$ is given by $e^{\tau(v-1)}$, while that of a zero-truncated Poisson (ZTP) distribution with parameter τ by

$$\frac{e^{\tau v} - 1}{e^{\tau} - 1}$$

whose mean and variance equal

$$\frac{\tau e^{\tau}}{e^{\tau} - 1} \quad \text{and} \quad \frac{\tau e^{\tau}(e^{\tau} - 1 - \tau)}{(e^{\tau} - 1)^2},$$

respectively. When $\tau := \log 2$, these become $2\log 2$ and $2(\log 2)(1 - \log 2)$, respectively. Also $\mathscr{N}(0,1)$ denotes the standard normal distribution. The notation $X_n \xrightarrow{d} X$ means convergence in distribution.

7.1.1. *Limit theorems.*

Theorem 21 (Statistics on Λ-row-Fishburn matrices)**.** *Assume that $\Lambda(z)$ is analytic at $z = 0$ with $\lambda_1 > 0$. Then in a random matrix (under the uniform distribution assumption on the set of Λ-row-Fishburn matrices of the size n),*

(i) the size X_n of the first row is distributed asymptotically as zero-truncated Poisson with parameter $\log 2$*:*

$$X_n \xrightarrow{d} \mathrm{ZTP}(\log 2),$$

(ii) the size Y_n of the diagonal (or the last column) is asymptotically normally distributed with mean and variance both asymptotic to $\log n$*,*

$$\frac{Y_n - \log n}{\sqrt{\log n}} \xrightarrow{d} \mathscr{N}(0,1), \quad \textit{and} \tag{7.1}$$

(iii) for the number Z_n of 1s, if $\lambda_2 > 0$, then

$$\frac{n - Z_n}{2} \xrightarrow{d} \text{Poisson}(\tau) \quad \text{with} \quad \tau := \frac{\lambda_2 \pi^2}{12\lambda_1^2}, \tag{7.2}$$

and $\mathbb{P}(Z_n = n) \to 1$ *if* $\lambda_2 = 0$.

For the diagonal size, we can also express the asymptotic distribution as $Y_n \sim \text{Poisson}(\log n)$, which implies (7.1). Finer approximations are given in (7.4) and (7.5).

Proof. (i) For the first row size X_n, we begin with the generating function (see (2.7))

$$f_X(z, v) := \sum_{k \geqslant 0} \big(\Lambda(vz)^{k+1} - 1\big) \prod_{1 \leqslant j \leqslant k} \big(\Lambda(z)^j - 1\big).$$

By applying (5.5) to $(d(z), e(z)) := (\Lambda(vz), \Lambda(z))$ and to $(d(z), e(z)) := (1, \Lambda(z))$, we deduce that

$$\mathbb{E}\big(v^{X_n}\big) := \frac{[z^n] f_X(z, v)}{a_n} \simeq 2^v - 1,$$

holding uniformly for bounded $v = O(1)$. This asymptotic estimate holds *a priori* pointwise for each finite $v \neq 0$, but the same proof gives indeed the uniformity of the error term in v when $v = O(1)$ and v stays away from zero. To include $v = 0$, we observe that $\mathbb{E}\big(v^{X_n}\big)$ is a polynomial without constant term (or equal to zero when $v = 0$) when $n \geqslant 1$; since the right-hand side also equals zero when $v = 0$, we conclude by analyticity the uniform bound in the region $v = O(1)$. This implies the convergence in distribution to the zero-truncated Poisson (ZTP) law with parameter $\log 2$.

(ii) Consider now the generating polynomial for the diagonal size Y_n

$$[z^n] f_Y(z, v) := [z^n] \sum_{k \geqslant 1} \prod_{1 \leqslant j \leqslant k} \big(\Lambda(vz)\Lambda(z)^{j-1} - 1\big).$$

The generating function is not of the form (5.1), but observe that

$$\Lambda(vz) = \Lambda(z)^v \big(1 + O(|z|^2)\big),$$

when $|z|$ is small. Then, when $k \asymp n$ and $|z| \asymp n^{-1}$ (taking logarithm and estimating the sum of errors), we have

$$\prod_{1 \leqslant j \leqslant k} \big(\Lambda(vz)\Lambda(z)^{j-1} - 1\big) = \left(\prod_{1 \leqslant j \leqslant k} \big(\Lambda(z)^{j+v-1} - 1\big) \right) \big(1 + O\big(|z| \log k\big)\big), \tag{7.3}$$

and we are in a position to apply Theorem 18, giving

$$[z^n] f_Y(z, v) = c(v) \rho^n n^{n+v-\frac{1}{2}} \big(1 + O\big(n^{-1} \log n\big)\big),$$

where

$$(c(v), \rho) := \left(\frac{\sqrt{\pi}}{\Gamma(v)} \Big(\frac{12\lambda_1}{\pi^2} \Big)^v e^{\frac{\pi^2}{12}\left(\frac{\lambda_2}{\lambda_1^2} - \frac{1}{2}\right)}, \frac{12\lambda_1}{e\pi^2} \right),$$

uniformly for bounded $v = O(1)$; note that $[z^n]f_Y(z,v)$ is again a polynomial without constant term, so the estimate also holds when $v = 0$ because $c(v) \to 0$ as $v \to 0$. Accordingly, the probability generating function of Y_n satisfies

$$\mathbb{E}\big(v^{Y_n}\big) = \frac{[z^n]f(z,v)}{a_n} = \frac{1}{\Gamma(v)}\left(\frac{12}{\pi^2}\right)^{v-1} e^{(v-1)\log n}\big(1 + O\big(n^{-1}\log n\big)\big),$$

uniformly for bounded $v = O(1)$. This is of the form of Quasi-Powers (see [20, 27]), and we then deduce the asymptotic normality of Y_n with optimal convergence rate:

$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\Big(\frac{Y_n - \log n}{\sqrt{\log n}} \leqslant x\Big) - \Phi(x)\right| = O\big((\log n)^{-\frac{1}{2}}\big), \tag{7.4}$$

together with the asymptotic approximations to the mean and the variance:

$$\begin{aligned} \mathbb{E}(Y_n) &= \log n + \gamma + \log\tfrac{12}{\pi^2} + O\big(n^{-1}\log n\big), \\ \mathbb{V}(Y_n) &= \log n + \gamma - \tfrac{\pi^2}{6} + \log\tfrac{12}{\pi^2} + O\big(n^{-1}(\log n)^2\big), \end{aligned} \tag{7.5}$$

where γ denotes the Euler-Mascheroni constant and $\Phi(x)$ denotes the distribution function of the standard normal distribution. For other types of Poisson approximation, see [29].

(iii) Applying the same proof of Theorem 18 to the generating function (2.9) for the number of 1s gives

$$[z^n]\sum_{k\geqslant 1}\prod_{1\leqslant j\leqslant k}\big((\Lambda(z) + \lambda_1(v-1)z)^j - 1\big) \simeq c(v)v^n\rho^n n^{n+\frac{1}{2}},$$

uniformly for bounded v, where $(c(v),\rho) := \Big(\frac{12}{\pi^{3/2}}\, e^{\frac{\pi^2}{12}\left(\frac{\lambda_2}{\lambda_1^2 v^2} - \frac{1}{2}\right)}, \frac{12\lambda_1 v}{e\pi^2}\Big)$. This implies that if $\lambda_2 > 0$, then

$$\mathbb{E}\big(v^{\frac{1}{2}(n-Z_n)}\big) \simeq e^{\tau(v-1)}, \quad \text{with} \quad \tau := \frac{\pi^2\lambda_2}{12\lambda_1^2}, \tag{7.6}$$

and we then obtain the limit Poisson distribution with parameter τ. If $\lambda_2 = 0$, then $\mathbb{E}\big(v^{n-Z_n}\big)$ tends to 1, a Dirac distribution. Furthermore, by the uniformity of (7.6) and Cauchy's integral representation, we obtain (7.2); see [27].

Similarly, for the number $Z_n^{[2]}$ of 2s, we use the generating function

$$\sum_{k\geqslant 1}\prod_{1\leqslant j\leqslant k}\big(\big(\Lambda(z) + \lambda_2(v-1)z^2\big)^j - 1\big),$$

and deduce that $\mathbb{E}\big(v^{Z_n^{[2]}}\big) \simeq e^{\tau(v-1)}$, with the same τ as in (7.2).

□

Stronger results such as local limit theorems can also be derived; see [27] for more information.

7.1.2. *Applications.* Consider first the case of primitive row-Fishburn matrices with $\Lambda = \{0, 1\}$. Then by Theorem 21, we see that in a random primitive Fishburn matrix the first row size is asymptotically ZTP(log 2) distributed, the diagonal is asymptotically normal, while the number of 1 is obviously the same as the size of the matrix. In particular, the distribution of the diagonal size corresponds to sequence A182319.

On the other hand, when $\Lambda(z) := \frac{1}{1-z}$, we have very similar behaviors for the sizes of the first row and the diagonal, but the number Z_n of 1s is asymptotically Poisson:

$$\mathbb{P}(n - Z_n = 2k) \to \tfrac{\tau^k}{k!}\, e^{-\tau}, \quad \text{with} \quad \tau = \tfrac{\pi^2}{12} \text{ for } k = 0, 1, \ldots.$$

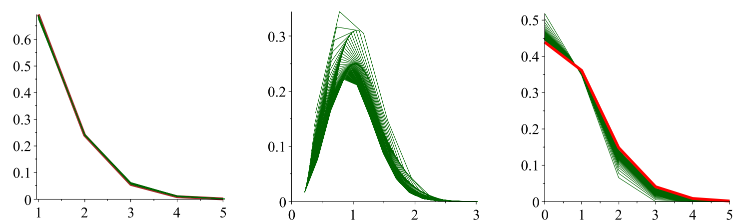


Figure 7.1. The histograms of X_n, Y_n and Z_n in the case of row-Fishburn matrices ($\Lambda(z) = \frac{1}{1-z}$) for $n = 6, \ldots, 50$ (see Theorem 21): $\mathbb{P}(X_n = k)$ (left), $\mathbb{P}(Y_n = \lfloor t\mu_n \rfloor)$ (middle), and $\mathbb{P}(n - Z_n = 2k)$ (right), where $\mu_n = \mathbb{E}(Y_n)$. Their convergence to ZTP, normal and Poisson is visible in each case, as well as the corresponding convergence rate.

7.2. **Statistics on Fishburn matrices.** We consider random Λ-Fishburn matrices in this subsection. By Proposition 1, the number of Λ-Fishburn matrices of size n is given by (see (2.5))

$$a_n := [z^n] \sum_{k\geqslant 0} \prod_{1\leqslant j\leqslant k} \big(1 - \Lambda(z)^{-j}\big),$$

and an asymptotic approximation is already derived in Corollary 20.

7.2.1. *Limit theorems.*

Theorem 22. *Assume that $\Lambda(z)$ is analytic at $z = 0$ with $\lambda_1 > 0$ and that all Λ-Fishburn matrices of size n are equally likely to be selected. Then in a random matrix, the size X_n of the first row (or the last column) and the diagonal size Y_n are both asymptotically normally distributed with logarithmic mean and variance in the following sense*

$$\frac{X_n - \log n}{\sqrt{\log n}} \xrightarrow{d} \mathscr{N}(0,1), \quad \textit{and} \quad \frac{Y_n - 2\log n}{\sqrt{2\log n}} \xrightarrow{d} \mathscr{N}(0,1), \tag{7.7}$$

and if $\lambda_2 > 0$, then the number Z_n of 1s is asymptotically Poisson distributed

$$\frac{n - Z_n}{2} \xrightarrow{d} \text{Poisson}(\tau) \quad \textit{with} \quad \tau := \frac{\lambda_2 \pi^2}{6\lambda_1^2}, \tag{7.8}$$

otherwise, $\lambda_2 = 0$ implies that $\mathbb{P}(Z_n = n) \to 1$.

Proof. (i) We begin with the generating function (see (2.11)) for the first row size

$$f_X(z,v) := \Lambda(vz)\sum_{k\geqslant 0}\Lambda(z)^k\prod_{1\leqslant j\leqslant k}\Big(\big(\Lambda(vz)\Lambda(z)^{j-1}-1\big)\big(\Lambda(z)^j-1\big)\Big).$$

By (7.3) and the expansion $\Lambda(vz) = 1 + O(|z|)$ for small $|z|$, we have

$$f_X(z,v) = \left(\sum_{k\geqslant 0}\Lambda(z)^k\prod_{1\leqslant j\leqslant k}\Big(\big(\Lambda(z)^{j+v-1}-1\big)\big(\Lambda(z)^j-1\big)\Big)\right)\big(1+O(|z|\log n)\big),$$

when $|z| \asymp n^{-1}$.

Similar to Theorem 18, we first derive, by the same methods used in the proof of Proposition 17, that

(7.9)
$$[z^n]\sum_{k\geqslant 0}e^{kz}\prod_{1\leqslant j\leqslant k}\big(e^{(j+\omega)z}-1\big)\big(e^{jz}-1\big) \simeq c_0(\omega)\rho^n n^{n+\omega+1},$$

where

$$(c_0(\omega),\rho) := \left(\frac{2\sqrt{6}}{\Gamma(1+\omega)}\Big(\frac{6}{\pi^2}\Big)^{1+\omega}, \frac{6}{e\pi^2}\right).$$

Briefly, α is almost 2 in the proof of Proposition 17, and the largest terms occur when $k \sim \mu n$ and $n|z| \sim \xi$ with (μ,ξ) as in (3.11), so that e^{kz} contributes an extra factor 2.

We now make the change of variables $\Lambda(z) = e^y$, and follow the same proof procedure of Theorem 18, yielding

$$[z^n]\sum_{k\geqslant 0}\Lambda(z)^k\prod_{1\leqslant j\leqslant k}\Big(\big(\Lambda(z)^{j+v-1}-1\big)\big(\Lambda(z)^j-1\big)\Big) \simeq c(v)\rho^n n^{n+v},$$

where

$$(c(v),\rho) := \left(\frac{2\sqrt{6}}{\Gamma(v)}\Big(\frac{6}{\pi^2}\Big)^{v} e^{\frac{\pi^2}{12}\left(\frac{\lambda_2}{\lambda_1^2}-\frac{1}{2}\right)}, \frac{6\lambda_1}{e\pi^2}\right).$$

We then deduce that

$$\mathbb{E}\big(v^{X_n}\big) = \frac{1}{\Gamma(v)}\Big(\frac{6}{\pi^2}\Big)^{v-1} n^{v-1}\big(1+O\big(n^{-1}\log n\big)\big),$$

uniformly for $v = O(1)$, and the asymptotic normality of X_n then follows again from the Quasi-Powers theorem [20, 27] or a standard characteristic function argument. Finer results such as (7.4) and (7.5) can also be derived.

(ii) For the size of the diagonal Y_n, we now have the generating function (see (2.12))

$$f_Y(z,v) := \Lambda(vz)\sum_{k\geqslant 0}\Lambda(z)^k\prod_{1\leqslant j\leqslant k}\big(\Lambda(vz)\Lambda(z)^{j-1}-1\big)^2.$$

By (7.3), the same arguments used in (i) for X_n and Theorem 18, we deduce that

$$[z^n]f_Y(z,v) = c(v)\rho^n n^{n+2v-1}\big(1+O\big(n^{-1}\log n\big)\big),$$

where

$$(c(v),\rho) := \left(\frac{2\sqrt{6}}{\Gamma(v)^2}\Big(\frac{6}{\pi^2}\Big)^{2v-1} e^{\frac{\pi^2}{12}\left(\frac{\lambda_2}{\lambda_1^2}-\frac{1}{2}\right)}, \frac{6\lambda_1}{e\pi^2}\right).$$

It follows that

$$\mathbb{E}\big(v^{Y_n}\big) = \frac{1}{\Gamma(v)^2}\Big(\frac{6}{\pi^2}\Big)^{2(v-1)} n^{2(v-1)}\big(1+O\big(n^{-1}\log n\big)\big),$$

uniformly for $v = O(1)$. The asymptotic normality then follows from Quasi-Powers Theorem.

(iii) Since $\lambda_1 > 0$, we can apply Theorem 18 to the generating function (2.13) for the number Z_n of 1s, which is

$$f_Z(z,v) := \sum_{k\geqslant 0}(\Lambda(z)+\lambda_1(v-1)z)^{k+1}\prod_{1\leqslant j\leqslant k}\big((\Lambda(z)+\lambda_1(v-1)z)^j-1\big)^2,$$

and we deduce that $\mathbb{E}\big(v^{\frac{1}{2}(n-Z_n)}\big) \simeq e^{\tau(v-1)}$, where $\tau := \frac{\pi^2\lambda_2}{6\lambda_1^2}$, which leads to a degenerate limit law when $\lambda_2 = 0$ and a Poisson limit law otherwise. The number of 2s follows the same law.

□

The most widely studied parameter is the size X_n of the first row of uniformly random Fishburn matrices, i.e., the case $\Lambda(z) = (1-z)^{-1}$. It appeared in Stoimenow's study [41] on chord diagrams, and later examined by Zagier in [45]. Then the limiting distribution of X_n was raised as an open question in [8, 31]. The generating function $f_X(z,v)$ for the first row size has been derived in several papers; see, for example, [4, 6, 21, 31, 44], A175579 and Section 2 for several other quantities with the same distribution as X_n. See also Table 6 and Figure 7.2 for the distribution of small n and graphical renderings.

$n\backslash k$	1	2	3	4	5	6	7
1	1						
2	1	1					
3	2	2	1				
4	5	6	3	1			
5	15	21	12	4	1		
6	53	84	54	20	5	1	
7	217	380	270	110	30	6	1

$n\backslash k$	1	2	3	4	5	6	7
1	1						
2	0	2					
3	0	1	4				
4	0	2	5	8			
5	0	5	14	18	16		
6	0	15	47	67	56	32	
7	0	53	183	287	267	160	64

Table 6. The number of Fishburn matrices of size n with first row size equal to k (left) and the diagonal size to k (right) for $n = 1,\dots,7$. The table on the left corresponds to A175579.

The mean and the variance of X_n satisfy

$$\mathbb{E}(X_n) = \log n + \gamma - \log\tfrac{\pi^2}{6} + O\big(n^{-1}\log n\big),$$
$$\mathbb{V}(X_n) = \log n + \gamma - \tfrac{\pi^2}{6} - \log\tfrac{\pi^2}{6} + O\big(n^{-1}(\log n)^2\big).$$

8. A framework for matrices without 1s and self-dual matrices

We discuss in this section the extension to the situation when $e_1 = 0$ and $e_2 > 0$ of the general framework (5.1). The general asymptotic expressions (5.5) and (5.6) certainly fail in such a case as the leading constant involves e_1 in the denominator.

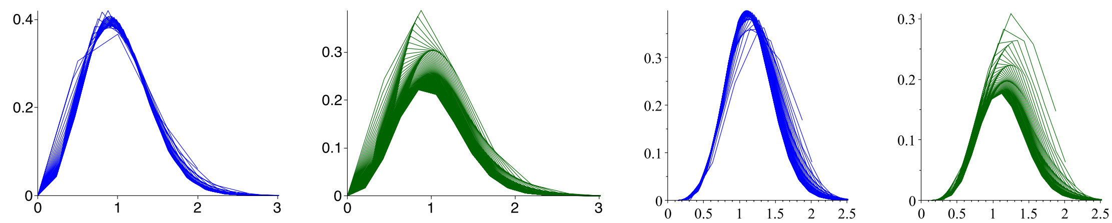


Figure 7.2. The histograms of X_n and Y_n (Fishburn matrices) for $n = 6,\dots,100$ (see Theorem 22): $\sigma_n(X)\mathbb{P}(X_n = \lfloor t\mu_n(X)\rfloor)$ (first), $\mathbb{P}(X_n = \lfloor t\mu_n(X)\rfloor)$ (second), $\sigma_n(Y)\mathbb{P}(Y_n = \lfloor t\mu_n(Y)\rfloor)$ (third), $\mathbb{P}(Y_n = \lfloor t\mu_n(Y)\rfloor)$ (fourth), where $\mu_n(W)$ and $\sigma_n^2(W)$ denote the corresponding mean and variance of W_n, respectively.

In addition to providing a better understanding of Fishburn matrices in more general situations, our consideration of (5.1) with $e_1 = 0$ and $e_2 > 0$ was also motivated by asymptotic enumeration of the self-dual Fishburn matrices, a conjecture raised by Jelínek (Conjecture 5.4 of [31]). In particular, the asymptotic approximations of non-primitive and primitive self-dual Fishburn matrices (given in (8.5) and (8.4)) will follow readily from our general result Theorem 23 or Corollary 26. Furthermore, as in Sections 6 and 7, our framework will be equally useful in characterizing the asymptotic distributions of a few statistics in random self-dual Fishburn matrices, which we briefly explore in this section.

While most proofs in this section follow similar ideas to the ones we employed in Section 3–7, the technical details in these proofs are more involved with generally lengthier expressions. Thus we will indicate the major differences.

8.1. **Asymptotics of (5.1) with $e_1 = 0$ and $e_2 > 0$.**

Theorem 23. *Assume $\alpha \in \mathbb{Z}^+$ and $\omega_0, \omega \in \mathbb{C}$. Given any two functions $e(z) := 1 + \sum_{j\geqslant 1} e_j z^j$ and $d(z) := 1 + \sum_{j\geqslant 1} d_j z^j$ that are analytic at $z = 0$, satisfying $e_1 = 0$, $e_2 > 0$, and*

$$\alpha e_3 \pi^2 + 12 d_1 e_2 \log 2 > 0, \tag{8.1}$$

we have

$$[z^n] \sum_{k\geqslant 0} d(z)^{k+\omega_0} \prod_{1\leqslant j\leqslant k} \left(e(z)^{j+\omega} - 1\right)^\alpha = c e^{\beta\sqrt{n}} \rho^{\frac12 n} n^{\frac12(n+\alpha)+\alpha\omega}\left(1 + O\left(n^{-\frac12}\right)\right), \tag{8.2}$$

the O-term holding uniformly for bounded ω_0 and ω, where $\beta := \frac{\sqrt{6}d_1\log 2}{\sqrt{e_2\alpha}\,\pi} + \frac{\sqrt{\alpha}\,e_3\pi}{2\sqrt{6}\,e_2^{3/2}}$, $\rho := \frac{6e_2}{e\pi^2\alpha}$, and

$$c := \frac{\sqrt{3}}{\sqrt{2}\,\alpha\pi}\left(\frac{1}{\Gamma(1+\omega)}\sqrt{\frac{12}{\alpha\pi}}\left(\frac{6}{\alpha\pi^2}\right)^\omega\right)^\alpha 2^{-\frac{d_1^2}{2e_2} - \frac{3d_1e_3}{4e_2^2} + \frac{d_2}{e_2}} e^{-\frac{d_1^2}{4\alpha e_2} - \frac{\alpha\pi^2}{12}\left(\frac{7e_3^2}{8e_2^3} - \frac{e_4}{e_2^2} + \frac12\right) + \frac{3d_1^2}{2e_2\alpha\pi^2}(\log 2)^2}.$$

Note specially the change of the dominant exponential part $\rho^{\frac12 n} n^{\frac12(n+\alpha)+\alpha\omega}$ in (8.2), as well as the presence of the extra factor $e^{\beta\sqrt{n}}$ when compared to (1.1) and (1.4). On the other hand, $\beta > 0$ is equivalent to the condition (8.1). When $\beta = 0$ (and $e_2 > 0$), asymptotic periodicities emerge (depending on the parity of n), which complicate the corresponding expressions. Instead of formulating a general heavy result, we will content ourselves with the study of Fishburn matrices with $\lambda_{2i-1} = 0$ for $1 \leqslant i \leqslant m$, but $\lambda_2, \lambda_{2m+1} > 0$ in Section 8.4.

The proof of Theorem 23 is similar to that of Theorem 5 and Theorem 18, beginning first with the corresponding exponential version and following by the change of variables $e(z) = e^{y^2}$ (locally invertible).

Proposition 24. *For large n, $\alpha \in \mathbb{Z}^+$, and $\omega \in \mathbb{C}$,*

$$[z^n] \sum_{k\geqslant 0} e^{kz} \prod_{1\leqslant j\leqslant k} \big(e^{(j+\omega)z^2} - 1\big)^\alpha = c e^{\beta\sqrt{n}} \rho^{\frac{1}{2}n} n^{\frac{1}{2}(n+\alpha)+\alpha\omega}\big(1 + O\big(n^{-\frac{1}{2}}\big)\big), \tag{8.3}$$

the O-term holding uniformly for bounded ω, where $\beta := \frac{\sqrt{6}\log 2}{\sqrt{\alpha\pi}}$, and

$$(c, \rho) := \left(\tfrac{\sqrt{3}}{\sqrt{2}\,\alpha\pi}\left(\tfrac{1}{\Gamma(1+\omega)}\sqrt{\tfrac{12}{\alpha\pi}}\left(\tfrac{6}{\alpha\pi^2}\right)^\omega\right)^\alpha e^{-\frac{1}{4\alpha}+\frac{3}{2\alpha\pi^2}(\log 2)^2}, \tfrac{6}{e\pi^2\alpha}\right).$$

Proof. (Sketch) Similar to the proof of Proposition 17; note that $A_k(z^2) = \prod_{1\leqslant j\leqslant k}\big(e^{jz^2} - 1\big)$ has the same order of magnitude as $A_k(r^2)$ when $z = re^{i\theta}$ with $\theta \sim \pi$, but due to the presence of e^{kz}, the corresponding Cauchy integral remains asymptotically negligible. □

Note that the proof of Theorem 23 can be extended to the situation when m ($m \geqslant 2$) is the smallest nonzero entry, that is, $\lambda_j = 0$ for $1 \leqslant j < m$ and $\lambda_m > 0$, $m \geqslant 2$.

8.2. Self-dual Λ-Fishburn matrices with $\lambda_1 > 0$.

We now consider general self-dual Λ-Fishburn matrices with $\lambda_1 > 0$.

Lemma 25. *The generating function for self-dual Λ-Fishburn matrices is given by (z marking the matrix size)*

$$\sum_{k\geqslant 0} \Lambda(z)^{k+1} \prod_{1\leqslant j\leqslant k} \big(\Lambda\big(z^2\big)^j - 1\big).$$

This lemma is a direct consequence of the case $\Lambda = \mathbb{Z}_{\geqslant 0}$ given in [31].

Corollary 26. *Assume that $\Lambda(z)$ is analytic at $z = 0$ and $\lambda_1 > 0$. Then the number of self-dual Λ-Fishburn matrices of size n satisfies*

$$[z^n] \sum_{k\geqslant 0} \Lambda(z)^{k+1} \prod_{1\leqslant j\leqslant k} \big(\Lambda(z^2)^j - 1\big) = c e^{\beta\sqrt{n}} \rho^{\frac{1}{2}n} n^{\frac{1}{2}(n+1)}\big(1 + O\big(n^{-\frac{1}{2}}\big)\big),$$

where $\beta := \frac{\sqrt{6\lambda_1}}{\pi}\log 2$, and $(c, \rho) := \left(\frac{3\sqrt{2}}{\pi^{3/2}}\, 2^{\frac{\lambda_2}{\lambda_1}-\frac{\lambda_1}{2}} e^{-\frac{\lambda_1}{4}-\frac{\pi^2}{24}+\frac{\pi^2\lambda_2}{12\lambda_1^2}+\frac{3\lambda_1}{2\pi^2}(\log 2)^2}, \frac{6\lambda_1}{e\pi^2}\right)$.

Proof. Condition (8.1) holds because $d_1 > 0$ and $e_3 = 0$. Apply Theorem 23 with $\omega_0 = \alpha = 1$, $\omega = 0$, $d_1 = e_2 = \lambda_1$, $d_2 = e_4 = \lambda_2$. □

This implies that if λ_1 is fixed, then no matter how many copies of other positive integers are used as entries, the resulting asymptotic count of self-dual matrices of large size differs only in the leading constant, provided that $\Lambda(z)$ is analytic at the origin.

Corollary 27. *(Conjecture 5.4 of [31]) The number of primitive self-dual and self-dual Fishburn matrices of size n are asymptotically given by*

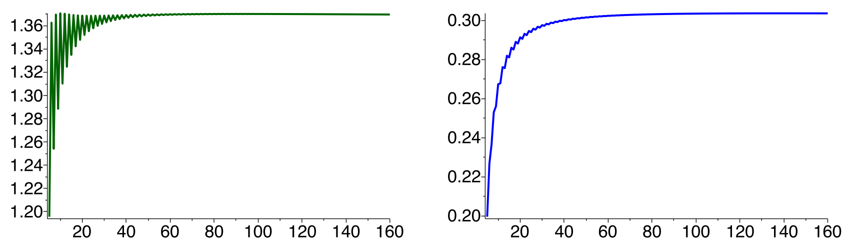
$$[z^n]\sum_{k\geqslant 0}(1+z)^{k+1}\prod_{1\leqslant j\leqslant k}\big((1+z^2)^j-1\big) = \frac{c}{2}\, e^{-\frac{\pi^2}{12}} e^{\beta\sqrt{n}}\rho^{\frac{1}{2}n} n^{\frac{1}{2}(n+1)}\big(1+O\big(n^{-\frac{1}{2}}\big)\big), \tag{8.4}$$

$$[z^n]\sum_{k\geqslant 0}(1-z)^{-k-1}\prod_{1\leqslant j\leqslant k}\big((1-z^2)^{-j}-1\big) = c e^{\beta\sqrt{n}}\rho^{\frac{1}{2}n} n^{\frac{1}{2}(n+1)}\big(1+O\big(n^{-\frac{1}{2}}\big)\big), \tag{8.5}$$

where $\beta := \frac{\sqrt{6}\log 2}{\pi}$ *and* $(c,\rho) := \Big(\frac{6}{\pi^{3/2}}\, e^{\frac{\pi^2}{24}-\frac{1}{4}+\frac{3}{2\pi^2}(\log 2)^2}, \frac{6}{e\pi^2}\Big)$.

Remark 2. *The constant* $c \approx 1.361951039$ *(see Figure 8.1) is given in an approximate numerical form in [31]. By comparing these estimates with (1.1), we see that the proportion of self-dual Fishburn matrices is asymptotically negligible (indeed factorially small).*

$$\frac{\text{LHS of (8.5)}}{e^{\beta\sqrt{n}}\rho^{\frac{1}{2}n}n^{\frac{1}{2}(n+1)}}\left(1-\frac{0.2}{\sqrt{n}}\right) \qquad \frac{\text{LHS of (8.4)}}{e^{\beta\sqrt{n}}\rho^{\frac{1}{2}n}n^{\frac{1}{2}(n+1)}}\left(1-\frac{1.5}{\sqrt{n}}\right)$$



$$\frac{6}{\pi^{3/2}}\, e^{\frac{\pi^2}{24}-\frac{1}{4}+\frac{3}{2\pi^2}(\log 2)^2} \approx 1.362 \qquad \frac{3}{\pi^{3/2}}\, e^{-\frac{\pi^2}{24}-\frac{1}{4}+\frac{3}{2\pi^2}(\log 2)^2} \approx 0.299$$

Figure 8.1. Numerical convergence of the two ratios $\frac{\text{LHS of (8.5)}}{e^{\beta\sqrt{n}}\rho^{\frac{1}{2}n}n^{\frac{1}{2}(n+1)}}$ and $\frac{\text{LHS of (8.4)}}{e^{\beta\sqrt{n}}\rho^{\frac{1}{2}n}n^{\frac{1}{2}(n+1)}}$ (with proper corrections for the O-terms) to their respective limit c.

We now examine the three statistics (first row-size, diagonal sum, and the number of 1s) on random self-dual Λ-Fishburn matrices, beginning with the corresponding bivariate generating functions. For convenience, we include the empty matrix with size 0.

Proposition 28 (Statistics on self-dual Λ-Fishburn matrices)**.** *For self-dual Λ-Fishburn matrices, we have the following bivariate generating functions with z marking the matrix size and v marking respectively*

(i) the size of the first row

$$\Lambda(vz)\sum_{k\geqslant 0}\Lambda(z)^k\prod_{1\leqslant j\leqslant k}\big(\Lambda\big(vz^2\big)\Lambda\big(z^2\big)^{j-1}-1\big), \tag{8.6}$$

(ii) the size of the diagonal

$$\Lambda(vz)\sum_{k\geqslant 0}\Lambda(z)^k\prod_{0\leqslant j< k}\big(\Lambda\big(v^2z^2\big)\Lambda\big(z^2\big)^{j}-1\big), \quad \textit{and} \tag{8.7}$$

(iii) the number of 1*s*

$$\sum_{k\geqslant 0}(\Lambda(z)+\lambda_1(v-1)z)^{k+1}\prod_{1\leqslant j\leqslant k}\big(\big(\Lambda(z^2)+\lambda_1(v^2-1)z^2\big)^j-1\big). \tag{8.8}$$

This is in analogy to Proposition 4, using the same ideas in [31] for counting self-dual matrices.

Theorem 29 (Statistics on self-dual Λ-Fishburn matrices)**.** *Assume that* $\Lambda(z)$ *is analytic at* $z=0$ *with* $\lambda_1>0$*. Then in a random matrix (under the uniform distribution assumption on self-dual* Λ*-Fishburn matrices of the size* n*), the size* X_n *of the first row (or the last column) and the half of the diagonal size* $\frac{1}{2}Y_n$ *both satisfy a central limit theorem with logarithmic mean and variance:*

$$\frac{X_n-\log n}{\sqrt{\log n}}\xrightarrow{d}\mathscr{N}(0,1),\quad \textit{and}\quad \frac{\frac{1}{2}Y_n-\log n}{\sqrt{\log n}}\xrightarrow{d}\mathscr{N}(0,1),$$

and for the number Z_n *of* 1*s, if* $\lambda_2>0$*, then* $n-Z_n$ *tends to the convolution of two Poisson distributions:*

$$n-Z_n\xrightarrow{d}2\text{Poisson}\big(\tfrac{\lambda_2}{\lambda_1}\log 2\big)*4\text{Poisson}\big(\tfrac{\lambda_2\pi^2}{12\lambda_1}\big),$$

while if $\lambda_2=0$*, then* $\mathbb{P}(Z_n=n)\to 1$*.*

Proof. (Sketch) The proofs for the random variables X_n and $\frac{Y_n}{2}$ rely on Theorem 23, in parallel of Theorem 22. For the number of 1s, Theorem 23 does not apply to (8.8) because $e_2=\lambda_1v^2$ is a complex number in general and $e_2>0$ may not hold. However, the proof there does apply by considering $e(z/\sqrt{e_2})$, similar to Theorem 18. The result is the same as if we apply formally Theorem 18 with $\omega_0=\alpha=1$, $\omega=0$, $d_1=\lambda_1 v$, $d_2=\lambda_2$, $e_2=\lambda_1v^2$, $e_3=0$, $e_4=\lambda_2$, yielding

$$\mathbb{E}\big(v^{n-Z_n}\big)=2^{\frac{\lambda_2}{\lambda_1}(v^2-1)}e^{\frac{\lambda_2\pi^2}{12\lambda_1^2}(v^4-1)}\big(1+O\big(n^{-\frac{1}{2}}\big)\big),$$

where the first term on the right-hand side is the probability generating function of two Poisson distributions if $\lambda_2>0$. The right-side becomes 1 when $\lambda_2=0$. ☐

8.3. **Asymptotics of Λ-Fishburn matrices whose smallest nonzero entry is** 2**.** We consider Fishburn matrices whose smallest nonzero entry is 2. We assume that there is at least one odd number in Λ, namely,

$$\lambda_{2k-1}=0,\ \text{for } 1\leqslant k\leqslant m\quad\text{and}\quad \lambda_2,\lambda_{2m+1}>0, \tag{8.9}$$

for $m\geqslant 1$. Otherwise, if Λ contains only even numbers, then, by dividing all entries by 2, the corresponding asymptotics and distributional properties can be dealt with by the same framework of Section 5. It turns out that $m=1$ (that is, $\lambda_1=0$ but $\lambda_2,\lambda_3>0$) and $m\geqslant 2$ have different asymptotic behaviors, and in the latter case the dependence on the parity of n is more pronounced, one technical reason being that the condition (8.1) fails when $m\geqslant 2$, and the odd case needs special treatment.

Lemma 30. *Given a formal power series* $B(z)=\sum_{n\geqslant 0}b_nz^n$ *with* $b_n\simeq c_0\rho_0^n n^{n+t}$*,* $\rho_0\neq 0$*, we have, for even* n*,*

$$[z^{\frac{1}{2}n}]e^{\beta nz}B(z)\simeq c\rho^{\frac{1}{2}n}n^{\frac{1}{2}n+t},\quad\textit{with}\quad (c,\rho):=\big(c_0 2^{-t}e^{\frac{2\beta}{e\rho_0}},\tfrac{1}{2}\rho_0\big).$$

Proof. Expand $e^{n\beta z}$ at $z = \frac{2}{e\rho_0 n}$, the asymptotic saddle-point of $z^{-\frac{1}{2}n}B(z)$, and estimate the error as in Section 4.2. ☐

Theorem 31 (Λ-Fishburn matrices with 2 as the smallest entries)**.** *Assume that Λ is a multiset of nonnegative integers satisfying (8.9) with $\Lambda(0) = 1$ and $\Lambda(z)$ analytic at $z = 0$. If $m = 1$, then the number of Λ-Fishburn matrices of size n satisfies*

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(1-\Lambda(z)^{-j}\big) = ce^{\beta\sqrt{n}}\rho^{\frac{1}{2}n}n^{\frac{1}{2}n+1}\big(1+O\big(n^{-\frac{1}{2}}\big)\big),$$

where $\beta := \frac{\lambda_3\pi}{2\sqrt{3}\,\lambda_2^{3/2}}$, *and* $(c,\rho) := \Big(\frac{3\sqrt{6}}{\pi^2}\,e^{\frac{\pi^2}{6}\left(\frac{\lambda_4}{\lambda_2^2}-\frac{1}{2}-\frac{7\lambda_3^2}{8\lambda_2^3}\right)}, \frac{3\lambda_2}{e\pi^2}\Big)$*; and if $m \geqslant 2$, then*

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(1-\Lambda(z)^{-j}\big) = \begin{cases} c'e^{\beta\sqrt{n}}\rho^{\frac{1}{2}n}n^{\frac{1}{2}n+1}\big(1+O\big(n^{-\frac{1}{2}}\big)\big), & \text{if } n \text{ is even;}\\ c_m e^{\beta\sqrt{n}}\rho^{\frac{1}{2}n}n^{\frac{1}{2}n-m+\frac{5}{2}}\big(1+O\big(n^{-\frac{1}{2}}\big)\big), & \text{if } n \text{ is odd,}\end{cases} \tag{8.10}$$

where ρ and β remain the same, $c' := \frac{6\sqrt{6}}{\pi^2}\,e^{\frac{\pi^2}{6}\left(\frac{\lambda_4}{\lambda_2^2}-\frac{1}{2}\right)}$, *and* $c_m := \frac{\sqrt{2}\pi^{2m-3}}{3^{m-2}}\cdot\frac{\lambda_{2m+1}}{\lambda_2^{m+1/2}}\,e^{\frac{\pi^2}{6}\left(\frac{\lambda_4}{\lambda_2^2}-\frac{1}{2}\right)}$.

Proof. (Sketch) When $m = 1$, apply Theorem 23 to the right-hand side of (2.5). When $m = 2$, following the proof of Theorem 23, we begin with the change of variables $\Lambda(z) = e^{y^2}$ and then apply Theorem 18 and Lemma 30 to prove (8.10). In particular, when $m \geqslant 2$, by splitting $\Lambda(z)$ into odd and even parts, using Lagrange's inversion formula in the form

$$[y^k]z = \frac{1}{k}[t^{k-1}]\Big(\frac{t}{\log\Lambda(t)}\Big)^k \qquad (k = 1, 2, \dots),$$

we deduce (8.10) when n is even; the expression of c_m then follows from that in the even case. See Theorem 33 of the first version of this paper on arXiv for details. ☐

In particular, the number of Fishburn matrices without occurrence of 1 as entries ($\Lambda = Z_{\geqslant 0}\setminus\{1\}$) satisfies

$$[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\left(1-\Big(\frac{1-z}{1-z+z^2}\Big)^j\right) = ce^{\beta\sqrt{n}}\rho^{\frac{1}{2}n}n^{\frac{1}{2}n+1}\big(1+O\big(n^{-\frac{1}{2}}\big)\big),$$

where $\beta := \frac{\pi}{2\sqrt{3}}$, and $(c,\rho) := \big(\frac{3\sqrt{6}}{\pi^2}\,e^{-\frac{\pi^2}{16}}, \frac{3}{e\pi^2}\big)$, which marks a significant difference with that containing 1 as entries, as given in (1.1). Similar behaviors are also exhibited in the asymptotics of row-Fishburn matrices without entry 1.

On the other hand, asymptotics of Λ-row-Fishburn matrices can be similarly treated, and exhibits a very similar behavior.

8.4. **Statistics on Λ-Fishburn matrices whose smallest nonzero entry is 2.** Based on the generating functions of Proposition 4, we now consider the behavior of a general random Λ-Fishburn matrix in which 2 is the smallest nonzero entry.

Theorem 32. *Assume that Λ is analytic at the origin and satisfies (8.9). Then in a random matrix (under the uniform distribution on the set of all Λ-Fishburn matrices of size n), the* *size X_n of the first row (or the last column) and the diagonal size Y_n are both asymptotically normally distributed in the following sense:*

$$\frac{X_n - \log n}{\sqrt{\log n}} \xrightarrow{d} \mathscr{N}(0,1), \quad \textit{and} \quad \frac{Y_n - 2\log n}{\sqrt{2\log n}} \xrightarrow{d} \mathscr{N}(0,1),$$

while the limiting distribution of the number Z_n of occurrences of 2 depends on m: if $m = 1$, then

$$\frac{\frac{1}{3}(n-2Z_n) - \tau\sqrt{n}}{\sqrt{\tau\sqrt{n}}} \xrightarrow{d} \mathscr{N}(0,1), \qquad \Big(\tau := \tfrac{\lambda_3\pi}{2\sqrt{3}\lambda_2^{3/2}}\Big), \tag{8.11}$$

and if $m \geqslant 2$, then

$$Z_n^* \xrightarrow{d} \mathrm{Poisson}\big(\tfrac{\lambda_4\pi^2}{6\lambda_2}\big), \quad \textit{with} \quad Z_n^* := \begin{cases} \frac{1}{2}\big(\frac{1}{2}n - Z_n\big), & \textit{if } n \textit{ is even;} \\ \frac{1}{2}\big(\frac{1}{2}(n-2m-1) - Z_n\big), & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. When $m = 1$, the proof relies on Theorem 23, following the same ideas used in the proof of Theorem 22, and when $m \geqslant 2$, the proof is similar to that of Theorem 31.

(i) Assume $m = 1$. For the first row sum, we have, by the generating function (2.11), the approximation (7.3) and a modification of the proof of Theorem 23,

$$[z^n]\Lambda(vz)\sum_{k\geqslant 0}\Lambda(z)^k \prod_{1\leqslant j\leqslant k}\Big(\big(\Lambda(vz)\Lambda(z)^{j-1}-1\big)\big(\Lambda(z)^j-1\big)\Big) = c(v)\rho^{\frac{1}{2}n}n^{\frac{1}{2}n+v}\big(1+O\big(n^{-\frac{1}{2}}\big)\big),$$

where $(c(v),\rho) := \Big(\frac{\sqrt{6}}{\Gamma(v)}\big(\frac{3}{\pi^2}\big)^v e^{\frac{\pi^2}{6}\big(\frac{\lambda_4}{\lambda_2^2}-\frac{7\lambda_3^2}{8\lambda_2^3}-\frac{1}{2}\big)}, \frac{3\lambda_2}{e\pi^2}\Big)$. Thus

$$\mathbb{E}\big(v^{X_n}\big) = \frac{1}{\Gamma(v)}\Big(\frac{3}{\pi^2}\Big)^{v-1} n^{v-1}\big(1+O\big(n^{-\frac{1}{2}}\big)\big), \tag{8.12}$$

uniformly for $v = O(1)$. Then the asymptotic normality (or Poisson($\log n$)) follows from the Quasi-Powers theorem. When $m \geqslant 2$, by the same procedure as in the proof of Theorem 31, we then deduce the same asymptotic approximation (8.12) when n is even. When n is odd, the corresponding asymptotic approximation differs by a factor of n^{-m} as in Theorem 31 but the resulting normalizing expression is still (8.12).

(ii) Very similarly, for the diagonal size, by applying Theorem 23 to the generating function (2.12), we deduce that

$$\mathbb{E}\big(v^{Y_n}\big) = \frac{1}{\Gamma(v)^2}\Big(\frac{3}{\pi^2}\Big)^{2(v-1)} n^{2(v-1)}\big(1+O\big(n^{-\frac{1}{2}}\big)\big),$$

uniformly for $v = O(1)$. The same expression remains true when $m \geqslant 2$ and the proof proceeds along the lines of that of Theorem 31.

(iii) Regarding the number of 2s, it is more involved. Consider first $m = 1$. Parallel to (2.13) for the number of 1s, we now have the generating function

$$\begin{aligned} &\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\big(1-\big(\Lambda(z)+\lambda_2(v-1)z^2\big)^{-j}\big) \\ &\quad = \sum_{k\geqslant 0}\big(\Lambda(z)+\lambda_2(v-1)z^2\big)^{k+1}\prod_{1\leqslant j\leqslant k}\big(\big(\Lambda(z)+\lambda_2(v-1)z^2\big)^j-1\big)^2. \end{aligned} \tag{8.13}$$

Then if $\lambda_3 > 0$, we get, by a similar modification of the proof of Theorem 23 (see Theorem 29), the Quasi-Powers approximation,

$$\mathbb{E}\left(v^{\frac{1}{2}n-Z_n}\right) = c(v)e^{\tau\sqrt{n}(v^{\frac{3}{2}}-1)}\left(1+O\left(n^{-\frac{1}{2}}\right)\right),$$

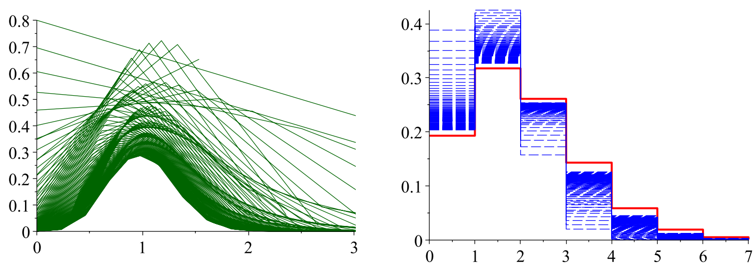
where τ is given in (8.11) and $c(v) := e^{-\frac{7\lambda_3^2\pi^2}{48\lambda_2^3}(v^3-1)+\frac{\lambda_4\pi^2}{6\lambda_2^2}(v^2-1)}$. The asymptotic normality then results from the Quasi-Powers theorem; see [20, 28]. Indeed, Z_n^* is asymptotically Poisson distributed with parameter $\tau\sqrt{n}$.

When $m \geqslant 2$, we obtain, by (8.13), the change of variables $\Lambda(z)+(\lambda_2-1)vz^2 = e^{y^2}$, and modifying the proof of (8.10) (see also the proof of Theorem 29),

$$\mathbb{E}\left(v^{\frac{1}{2}n-Z_n}\right) \sim \begin{cases} e^{\frac{\lambda_4\pi^2}{6\lambda_2^2}(v^2-1)}, & \text{if } n \text{ is even;} \\ v^{m+\frac{1}{2}}e^{\frac{\lambda_4\pi^2}{6\lambda_2^2}(v^2-1)}, & \text{if } n \text{ is odd.} \end{cases}$$

This proves the Poisson limit law.

□



$\Lambda(z) = 1+z^2+z^3$ $\qquad$ $\Lambda(z) = 1+z^2+z^4+z^5$

$\mathscr{N}\left(\frac{\pi\sqrt{n}}{2\sqrt{3}}, \frac{\pi\sqrt{n}}{2\sqrt{3}}\right)$ $\qquad$ $\text{Poisson}\left(\frac{\pi^2}{6}\right)$

Figure 8.2. Theorem 32: histograms of the number of 2s in two different compositions of random Λ-Fishburn matrices. Left: the distributions $\mathbb{P}\left(\frac{1}{3}\left(\frac{n}{2}-Z_n\right) = \lfloor x\mu_n\rfloor\right)$ *with* μ_n *denoting the exact mean, which is asymptotic to* $\frac{\pi\sqrt{n}}{2\sqrt{3}}$*; right:* $\mathbb{P}\left(Z_n^* = k\right)$*, where* $Z_n^* := \frac{1}{2}\left(\frac{n}{2}-Z_n\right)$ *when* n *is even, and* $Z_n^* := \frac{1}{2}\left(\frac{n}{2}-2-Z_n\right)$ *when* n *is odd, where the red line represents the corresponding Poisson distribution.*

For random Λ-row-Fishburn matrices, one can derive very similar types of results: zero-truncated Poisson with parameter $\frac{\lambda_1}{\lambda_2}\log 2$ for the first row size, $\mathscr{N}(\log n, \log n)$ for the diagonal size, and $\mathscr{N}\left(\tau\sqrt{n}, \tau\sqrt{n}\right)$ with $\tau := \frac{\lambda_3\pi}{2\sqrt{6}\lambda_2^{3/2}}$ or $\text{Poisson}\left(\frac{\lambda_4\pi^2}{12\lambda_2^2}\right)$ limit law when $m=1$ or $m \geqslant 2$, respectively, for the number of 2s.

9. Conclusions

Motivated by the asymptotic enumeration of and statistics on Fishburn matrices and their variants, we developed in this paper a saddle-point approach to compute the asymptotics of the coefficients of generating functions with a sum-of-product form, and applied it to several dozens of examples. The approach is not only useful for the usual large-n asymptotics but also effective in understanding the stochastic behaviors of random Fishburn matrices, with or without further constraints on the entries or on the structure of the matrices. In particular, we identified a simple yet general framework and showed its versatile usefulness in this paper. Many new asymptotic distributions of statistics on random matrices are derived in a systematic and unified manner, which in turn demand further structural interpretations; for example, since the normal approximations we derived in this paper can indeed all be approximated by Poisson distributions with parameters depending on n (equal to the asymptotic mean), a natural question is why Poisson laws with bounded or unbounded parameters are ubiquitous in the random Λ-Fishburn matrices.

Other frameworks will be examined in a follow-up paper. In addition to different sum-of-product patterns, we will also work out cases for which our approach in this paper does not directly apply. For example, we have not found transformations for the series $[z^n]\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\tanh(2jz)$, a special case of general theorems in [5], such that our saddle-point method works, although it is known (see [5]) that

$$\sum_{k\geqslant 0}\prod_{1\leqslant j\leqslant k}\tanh(2jz)=\sum_{n\geqslant 0}\frac{a_{2n+1}}{n!}\,z^n,\quad\text{and}\qquad \sum_{n\geqslant 0}\frac{a_{2n+1}}{(2n+1)!}\,z^{2n+1}=\tan(z),$$

where the a_{2n+1}'s are the tangent numbers. For similar pairs of series of this type, see [5, 26, 39].

Finally, the rank (or dimension) represents another important statistic on random matrices. In our recent paper [30], we proved that the dimension of a random Λ-Fishburn matrices follows a central limit theorem with linear mean and variance. Furthermore, the corresponding dual problem of size distribution under large dimension is also addressed and follows a quadratic type normal limit law. These answer two open problems in [8, 31] respectively. Interestingly, the saddle-point approach combined with a powerful transformation formula for q-series due to Andrews and Jelínek, is also useful in solving a conjecture of Stoimenow on Vassiliev invariants (see [30]).

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